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Analysis I — Problem Set 8

Issued: 29.10.07 Due: 06.11.07, noon

8.1. Continuous Maps

(a) Let

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \begin{cases} 0 & x \in \mathbb{R} \setminus \mathbb{Q} \\ 1/n & x = \frac{m}{n} \text{ coprime} \end{cases}$$

Prove that f is continuous exactly at all irrational points.

(b) Let

$$f : \mathbb{Q} \longrightarrow \mathbb{R}, \quad f(x) = \begin{cases} 0 & x > \sqrt{2} \\ 1 & x \leq \sqrt{2} \end{cases}$$

Prove that f is continuous.

8.2. Continuous functions attain average values. Let $f : (a, b) \to \mathbb{R}$ be continuous. Prove that for any choice of points $x_1, x_2, \ldots, x_n \in (a, b)$, there is an $x_0 \in (a, b)$ such that $f(x_0) = \frac{1}{n}(f(x_1) + f(x_2) + \ldots + f(x_n))$.

8.3. Continuity and the limes inferior. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $(x_n)_{n\in\mathbb{N}}$ be bounded.

(a) Show that $\liminf\limits_n f(x_n) \leq f(\liminf\limits_n x_n)$.

(b) Give an example for $\liminf\limits_n f(x_n) < f(\liminf\limits_n x_n)$.

8.4. Metric spaces Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. Prove that the following are equivalent:

(a) The function f is continuous.

(b) For all $x_0 \in X$ and for all sequences $(x_n)_{n\in\mathbb{N}}$ in X with $\lim\limits_{n\to\infty} x_n = x_0$ we have $\lim\limits_{n\to\infty} f(x_n) = f(x_0)$.

8.5. Uniform convergence of power series. Let $D = \{z \in \mathbb{C},\ |z| \leq 1\}$ be the unit ball in the complex plane and $\mathrm{X} = \{f : D \to \mathbb{C},\ f \text{ is bounded}\}$ and $d_u(f, g) = \sup\{|f(z) - g(z)|,\ z \in D\}$.

(a) Show that (X, d_u) is a metric space.

(b) Suppose that $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 1$. Show that $f \in \mathrm{X}$ and $f_N \in \mathrm{X}$, where $f_N : D \to \mathbb{C},\ z \mapsto \sum\limits_{n=0}^{N} a_n z^n$.

(c) Prove that $\lim\limits_{N\to\infty} f_N = f$ in the metric space (X, d_u).

Note: Recall that $f : A \to \mathbb{C}$ is bounded if there exists $M \in \mathbb{R}$ such that $|f(a)| \leq M$ for all $a \in A$.