Jacobs University
School of Engineering and Science
Götz Pfander, Alexandru Sava

Fall Term 2010
Monday, December 13^{th}
16:00 – 18:00

Analysis I — Final Exam

Notes: Sign your work to certify that you adhere to the academic Code of Honor to work independently. You can attempt each subproblem and use the results of the preceding subproblem. You may cite all results from class, homeworks, and examinations. You may use the first three chapters of the analysis script *All answers must be justified! Show all your work!*

Each problem is 50 points. Points achieved beyond 300 are counted as bonus points.

F.1. Fix $\alpha > 0$ and let $x_1 > \sqrt{\alpha}$. Let

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{\alpha}{x_n}\Big), \quad n \in \mathbb{N}.$$

(a) Show that $x_n > \sqrt{\alpha}$ for all $n \in \mathbb{N}$.

(b) Conclude that $\{x_n\}$ is monotonically decreasing.

(c) Show that $\{x_n\}$ converges to $\sqrt{\alpha}$.

(d) Set $e_n = x_n - \sqrt{\alpha}$. Show that

$$e_{n+1} \leq \frac{e_n^2}{2x_n}, \quad n \in \mathbb{N}.$$

(e) Show that

$$e_{n+1} \leq 2\sqrt{\alpha}\left(\frac{e_1}{2\sqrt{\alpha}}\right)^{2^n}, \quad n \in \mathbb{N}.$$

(f) For $\alpha = 3$ and $x_1 = 2$, how many terms do you need to compute to achieve $e_n \leq 10^{-3}$?

F.2. Give an example of each of the following (if it exists, if **not**, state why).

(a) A closed but not compact set of $\mathbb{R}$.

(b) A compact but not bounded subset of $\mathbb{R}$.

(c) An infinite set in $\mathbb{R}$ which does not have a cluster point.

(d) A metric space with a closed and bounded subset which is not complete.

(e) A compact metric space which is not complete.

(f) A topological space where all finite sets are open.

(g) A connected subset of $\mathbb{R}$.

(h) A subset of $\mathbb{R}$ that is not connected.

F.3. Let U, V be open intervals and $f : U \longrightarrow V$ be surjective and strictly monotonic increasing. Show that f and f^{-1} are continuous.

F.4. Show that the composition of uniformly continuous functions on a metric space X is uniformly continuous.

F.5. Let $f_k : [0,1] \longrightarrow \mathbb{R}$ be continuous functions and $f_k(x) \longrightarrow f_0(x)$ for all $x \in [0,1]$. Show that if $f_{k+1}(x) \geq f_k(x)$ for all $k \in \mathbb{N}$ and $x \in [0,1]$, then f_k converges uniformly to f_0.

F.6. Let $f : (0,1) \to \mathbb{R}$ be differentiable. Moreover, assume that f' is differentiable at a point $x_0 \in (0,1)$. Show that

$$f''(x_0) = \lim_{y \to 0} \frac{f(x_0 + y) + f(x_0 - y) - 2f(x_0)}{y^2}.$$

[Hint: L'Hospital.]

F.7. Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ and recall $\left(\begin{array}{c} \alpha \\ n \end{array} \right) = \frac{\alpha}{n} \cdot \frac{\alpha - 1}{n-1} \cdot \ldots \cdot \frac{\alpha - n + 3}{3} \cdot \frac{\alpha - n + 2}{2} \cdot \frac{\alpha - n + 1}{1}.$
Let $f_\alpha(x) = (1+x)^\alpha$.

(a) Show that the Taylor series of f_α at 0 is $T_\alpha(x) = \sum_{k=1}^{\infty} \left(\begin{array}{c} \alpha \\ n \end{array} \right) x^k.$

(b) Show that T_α converges absolutely for $|x| < 1$.

(c) Show that T_α converges to f_α for $x \in [0,1)$.