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Fall Term 2010
Thursday, October 21^{st}
11:15 – 12:30

Analysis I — Midterm Exam

Notes: Sign your work to certify that you adhere to the academic Code of Honor to work independently. You can attempt each subproblem and use the results of the preceding subproblem. You may cite all results from class, homeworks, and examinations. You may use the first two chapters of the analysis script *All answers must be justified! Show all your work!*

Each problem is 55 points. Do 4 out of 5. If you do 5, your best 4 problems will be counted.

MT.1. (a) Show that there are uncountably many intervals (a,b) in $\mathbb{R}$, $a \neq b$.

(b) Let E be an uncountable family of intervals. Prove that there exists at least two intervals in this family that overlap.

MT.2. Are $d(x,y) = \sqrt{|x-y|}$ and $\widetilde{d}(x,y) = |x-2y|$ metrics on $\mathbb{C}$?

MT.3. Prove or provide a counterexample for the following two statements.

(a) For real and bounded sequences (a_n) and (b_n) we have

$$\limsup_{n\to\infty} \max\{a_n, b_n\} = \max\{\limsup_{n\to\infty} a_n, \limsup_{n\to\infty} b_n\}$$

(b) For real and bounded sequences (a_n) and (b_n) we have

$$\limsup_{n\to\infty} \min\{a_n, b_n\} = \min\{\limsup_{n\to\infty} a_n, \limsup_{n\to\infty} b_n\}$$

MT.4. Suppose that the the power series $\sum_{n=0}^{\infty} b_n z^n$ converges for all $z \in \mathbb{C}$. Show that if $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = s$ exists, then $\sum_{n=0}^{\infty} a_n z^n$ converges for all $z \in \mathbb{C}$ as well. (60)

MT.5. Radii of convergence

Determine the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ if ...

(a) ...there are real numbers $L > 0$, α such that $\lim |a_n n^\alpha| = L$;

(b) ...there are real numbers $\alpha > 0$, L such that $\lim |a_n \alpha^n| = L$;

(c) ...$\lim_{n\to\infty} |a_n n!| = L$ for some $L > 0$.

MT.6. (a) Show that for $x_n \geq 0$ with $\lim_{n\to\infty} x_n = x$ we have $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

(b) Discuss the convergence of the sequence given by $a_1 = 1$, $a_{n+1} = \sqrt{2 + a_n}$ for $n \in \mathbb{N}$.

MT.7. Let (a_n) be a sequence of real numbers with $|a_n| \leq M$, $M > 0$, for all $n \in \mathbb{N}$. Show that

(a) $\sum_{n=1}^{\infty} a_n x^n$ converges for every $x \in (-1,1)$.

(b) If $a_1 > 0$, then $\sum_{n=1}^{\infty} a_n x^n \neq 0$ for all $x \in (-\frac{a_1}{2M}, 0) \cup (0, \frac{a_1}{2M})$.