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Analysis I —Assignement 10

10.1. *Derivative of functions*

Find the maximum domain of definition D for the following functions, argue that they are differentiable on D and then differentiate them :

(a) $f_1 : D \to \mathbb{R}, f_1(x) = x^x.$

(b) $f_2 : D \to \mathbb{R}, f_2(x) = \dfrac{\log(x^3 + x)}{x^2 + 1}$

(c) $f_3 : D \to \mathbb{R}, f_3(x) = \dfrac{\sin \sin x}{\cos \cos x}$

(d) $f_4 : D \to \mathbb{R}, f_4(x) = \dfrac{\log \log x}{\log x}$

(e) $f_5 : D \to \mathbb{R}, f_5(x) = x^3 \sin(\dfrac{1}{x})$ Hint: this function can be defined in 0.

NOTE: Highschool knowledge is assumed. Things like $(e^x)' = e^x, (\cos x)' = -\sin x etc.$ are assumed to be known.

10.2. *Differentiable function*

Recall the function : $f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ in lowest terms, } n > 0 \end{cases}$ is continuous at the irrationals and discontinuous at the rationals. Check if it is also differentiable on the irrationals.

Hint: Recall the properties of p_n, q_n for which $\lim_n \dfrac{p_n}{q_n} = r$ for some irrational r.

10.3. *Connected vs path connected*

(a) Let (X, d) be a metric space and $A \subset X$ a subset of X such that for any $p.q \in A$ there exist a continuous functions $f : [0, 1] \to A$ with $f(0) = p$ and $f(1) = q$. Such a subset A is called path-connected and f would be a path between p and q. Prove that A is connected.

(b) This naturally makes us think that maybe the converse is also true and that indeed every connected space is also path connected. This is however false. Prove that the following example is a good counterexample: let (X, d) be the plane with the normal matric ; now define $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ $g(x) = \begin{cases} 0, & \text{if } x = 0 \\ \sin \frac{1}{x} & \text{if } x \neq 0 \end{cases}$; let $A = \{(x, g(x)) : x \geq 0\}$ be the graph of g ; A is connected but not path connected. Argue why!

10.4. *An interesting property of connected sets*

Let (X, d) be a metric space and $A_i \subset X, i \in \overline{1 \ldots n}$ be connected subsets of X. If $\bigcap_k A_k \neq \emptyset$ then $\bigcup_k A_k$ is also connected.

Hint: Suppose the union is the set M and the intersection, the set N such that $M \subset X \cup Y$ with X, Y open, $X \cap Y = \emptyset$ and $N \neq \emptyset$. Let $N_X = N \cap X$, $N_Y = N \cap Y$. What can you say about N_X and N_Y ?

10.5. *Uniform convergence vs pointwise convergence*

Let $I = [0,1]$ and $I_0 = \overline{I} = [0,1]$. Define $f_n : I_0 \to \mathbb{R}$, $f_n(x) = \begin{cases} 0, \text{ if } 0 \leq x \leq 1 - \dfrac{1}{n} \\ nx - (n-1) \text{ if } 1 - \dfrac{1}{n} \leq x \leq 1 \end{cases}$.

(a) Show that that for every $x \in I_0$ the sequence $(f_n(x))_n$ converges to a point which we denote by $g(x)$.Argue that even if the $f_n's$ are continuous on I_0, g is not.

Hint: try to draw some of the $f_n's$ (these are piecewise linear functions) and see what happens especially at $x = 1$.

(b) Show that f_n does not uniformly converge on I_0.

Hint: uniform limit of continuous functions is continuous !

10.6. BONUS PROBLEM Weird function

Let $I = [0,1]$. Construct a function $f : I \to \mathbb{R}$ such that f is continuous but f is nowhere differetiable or in other words there is no $x \in I$ such that f is differetiable at x.