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# Analysis I —Assignement 11

**11.1.** *Monotonicity of differentiable functions*

Suppose $f'(x) > 0$ in (a,b).

(a) Prove that $f$ is strictly increasing on $(a, b)$.

(b) Let $g$ be the inverse function of $f$, e.g., $g(f(x)) = x$ for $x \in (a, b)$. Prove that $g$ is differentiable and that $g'(f(x)) = \frac{1}{f'(x)}$ for $x \in (a, b)$.

**11.2.** *Higher order derivatives*

Calculate $f^{(n)}(0)$ for $f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \begin{cases} e^{-\frac{1}{x^2}}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$ Recall that $f^{(n)}$ is the derivative of the function $f^{(n-1)}$, where $f^{(0)} = f$ and $f^{(1)} = f'$. (Tip: Use induction and L'Hospital.)

**11.3.** *Darboux property of derivatives*

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Let $a < b$ be two real numbers and suppose there are $x_0, y_0 \in \mathbb{R}$ such that $f'(x_0) = a$ and $f'(y_0) = b$. Prove that for every $c \in (a, b)$ there exist $t \in (\min(a, b), \max(a, b))$ such that $f'(t) = c$.

HINT : consider $g(x) = f(x) - cx$ on the interval $[\min(a, b), \max(a, b)]$

**11.4.** *Interesting consequence of derivatives*

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that $f$ is constant.

**11.5.** *Limits of functions*

Assume that $(f_n)$ is uniformly convergent on $A \subset \mathbb{R}$ to the function $f_0$. Assume moreover that $x_0$ is a cluster point of $A$ and, $\lim_{x \to x_0} f_n(x) = f(x_0)$ for $n \geq N$, $N \in \mathbb{N}$.

(a) Prove that $\lim\limits_{n \to \infty} \lim\limits_{x \to x_0} f_n(x) = \lim\limits_{x \to x_0} f_0(x)$.

(b) Prove that if $(f_n)$ is uniformly convergent on $(0, \infty)$ to the function $f_0$ and $\lim_{x \to \infty} f_n(x) = \alpha \in \mathbb{R}$ for $n \geq N$, $N \in \mathbb{N}$, then

$$\lim_{n \to \infty} \lim_{x \to \infty} f_n(x) = \lim_{x \to \infty} f(x)$$

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(c) Assume that $(f_n)$ is a sequence of functions which are differentiable on $(c, d)$, and let $[a, b] \subset (c, d)$. If $\sum_{n=1}^{\infty} f_n(x)$ converges at some $x_0 \in [a, b]$ and $\sum_{n=1}^{\infty} f'_n(x)$ converges uniformly on $[a, b]$, then $\sum_{n=1}^{\infty} f_n(x)$ converges to a differentiable function, and

$$\left( \sum_{n=1}^{\infty} f_n(x) \right)' = \sum_{n=1}^{\infty} f'_n(x).$$

(d) Show that $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$ is differentiable on $\mathbb{R}$.

**11.6. BONUS PROBLEM Sharkovskii's Theorem - Particular case**

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. A point $x \in \mathbb{R}$ is called $k$ - periodic if $f(f(\ldots(x)\ldots)) = x$ (where $f$ is composed with itself $k$ times) but $f(f(\ldots(x)\ldots)) \neq x$ ($f$ composed $i$ times) for all $1 \leq i < k$. Show that if $f$ admits a 3 - periodic point, then it admits a $k$ - periodic point for all $k \in N$.