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Analysis I —Assignement 12

12.1. *Some inequality*

Prove that:

(a) $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$

(b) $\sin x \geq x - \frac{x^3}{6}$ for all $x \in [0, \frac{\pi}{2}]$

Hint: Use Taylor expansion and group terms conveniently

12.2. *Some approximations using Taylor*

(a) Compute the following : e^{10} with 3 decimals ; $\cos 1$ with 4 decimals ; $\sqrt[5]{2}$ with 2 decimals.

(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is analytic such that $f(x) = f(-x)$ for all $x \in \mathbb{R}$ then:

- $f^{(n)}(0) = 0$ for all odd n.
- if P_n are the standard Taylor polynomials for f, we have $\lim_{x \to 0} \frac{f(x) - P_{2n}(x)}{x^{2n}} = 0$.

  NOTE : Normally $\lim_{x \to 0} \frac{f(x) - P_{2n}(x)}{x^{2n-1}} = 0$ which is weaker than the above version.

12.3. *Some bounds on f coming from bounds on f'*

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove that: If f' is bounded then there exist $A, B \in \mathbb{R}$ such that $|f(x)| \leq A + B|x|$ for all $x \in \mathbb{R}$.

12.4. *Some convexity property*

Recall the definition of convex functions from last homework. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and convex then f' is a monotonically increasing function. Conclude that if f is twice differentiable then f is convex iff $f'' > 0$.

12.5. *Drawing Taylor Polynomials*

This problem is meant to make you understand Taylor's theorem and the difference between infinitely differentiable functions and analytic functions. You are suppose to use some kind of math software : Mathematika, MATLAB, Octave etc. (the one that you feel most confortable with). If you are not familiar with any of these math software, you are required to look up tutorials on the web or ask for help.

Draw the Taylor polynomials P_n for $n \in \overline{1 \ldots 6}$ for the following functions :

(a) $f_1 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = \sin x$.

(b) $f_2 : \mathbb{R} \to \mathbb{R}$, $f_2(x) = e^x$.

(c) $f_3 : \mathbb{R} \to \mathbb{R}$, $f_3(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$.

(d) $f_4 : (0,1) \to \mathbb{R}$, $f_4(x) = \dfrac{1}{1-x}$.

For a certain f_i, put the corresponding plots into the same graphs, label each graph and explain what you observe.

12.6. BONUS PROBLEM An important property of analytic functions

Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Let $K \subset \mathbb{R}$ be a compact set. Suppose there exist a sequence $(x_n)_n \subset K$ such that $f(x_n) = 0$ for all $n \in \mathbb{N}$. Prove that $f = 0$.