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Analysis I — Problem Set 2

Issued: 14.09.04 Due: 20.09.04

2.1. Show rigorously that for every $n \in \mathbb{N}$, $(\mathbb{Z}_n, +)$ is an abelian group. (That includes showing that $+$ is a well-defined operation).

(Bonus problem) Define $[x] \cdot [y] := [xy]$. Is that a well-defined operation? For which n is $(\mathbb{Z}_n, +, \cdot)$ a field? Prove your assertions.

2.2. Prove Theorem 1.11 in the script.

2.3. Let $(F, +, \cdot)$ be a field.

(a) Show that 0 is uniquely determined: there is only one neutral element under addition.

(b) For every $x \in F$, $-x$ is uniquely determined: there is only one $y \in F$ with $x + y = 0$.

(c) Show that for every $x, y \in F$ we have $0x = 0$; $(-x)y = -(xy)$; $(-x)(-y) = xy$.

2.4. Show the following statements in an ordered field. Remember $x > y :\Leftrightarrow (x \geq y \wedge x \neq y)$.

(a) If $x > 0$ then $0 > -x$, and vice versa.

(b) If $x > 0$ and $z > y$, then $xz > xy$.

(c) If $0 > x$ and $z > y$ then $xy > xz$.

(d) If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.

(e) If $y > x > 0$ then $1/x > 1/y > 0$.

2.5. Show that $(\mathbb{R}, +)$ is an abelian group using the definition of real numbers as *Dedekind cuts* and the addition on the cuts as defined in class.

2.6. Let $q \in \mathbb{Q} \setminus \{0\}$ and $r \in \mathbb{R}$ irrational. Show that both $q + r$ and $q \cdot r$ are irrational.