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Analysis I —Assignement 3

3.1. *Supremum properties*

Prove or disprove the following of the supremum :

(a) for a nonempty subset of $\mathbb{R}$ which is bounded below :

$$\inf A = -\sup\{-A\} \text{ where } -A = \{-x|x \in A\}$$

(b) for two nonempty subsets of $\mathbb{R}$, call them A and B :

$$\sup(A+B) = \sup A + \sup B \text{ where } A+B = \{x+y|x \in A \text{ and } y \in B\}$$

(c)

$$\sup(A \cdot B) = \sup A \cdot \sup B \text{ where } A \cdot B = \{x \cdot y|x \in A \text{ and } y \in B\}$$

If any of the above is false, give a sufficient condition for it to hold.

3.2. *Rational Powers*

(a) Show that for every non-negative real number b and every natural number $n > 0$ there exist an unique non-negative real number a denoted $b^{\frac{1}{n}}$, such that $a^n = b$

(b) Let $m, p \in \mathbb{Z}$ and $n, q \in \mathbb{N}$ such that : $\frac{m}{n} = \frac{p}{q} = r$. For $b \in \mathbb{R}$, $b > 0$, set :

$$b^r = (b^m)^{\frac{1}{n}}$$

Show that b^r is well defined, i.e. show that $(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}$.

(c) Justify why it makes sense to define $b^{\frac{1}{m}}$, $n \in \{2k+1|k \in \mathbb{N} \cup \{0\}\}$, only for non-negative numbers b.
Hint: Consider the expression $(-8)^{\frac{1}{3}}$.

(d) Let $r, s \in \mathbb{Q}$. Show that : $b^{r+s} = b^r b^s$ for $b > 0$.

3.3. *Interesting property of the reals* Let $M \subset \mathbb{R}^+$ be uncountable. Show that for every $r \in \mathbb{R}$ there is a finite number of different elements $a_1, a_2, \ldots, a_n \in M$ such that :

$$\sum_{k=1}^{n} \geq r$$

Hint: Among the sets $M_N = \{a \in M | a \geq \frac{1}{N}\}$, $N \in \mathbb{N}$, there has to be an infinite one.

3.4. *Calculus with complex numbers*

Let $z = \frac{1+\sqrt{3}i}{2}$

(a) Determine $|z|, z^{-1}, \overline{z}, z^n$ for $n \in \mathbb{N}$.

(b) Let $T \subset \mathbb{C}$ be the triangle with vertices $a = 5, b = 6+i, c = 7$. What geometric shape do $z^n a, z^n b, z^n c$ form for any $n \in \mathbb{N}$? Draw them into the complex plane together with T and z.

(c) Calculate $(1+i)^n + (1-i)^n$ for any $n \in \mathbb{Z}$. Why is the result always a real number ?

3.5. *Square roots and complex numbers*

Show that every complex number $w = u + iv$ has exactly two square roots (with one exception).

Hint : Define $z = a + bi$ with $a, b = \sqrt{\frac{1}{2}(|w| \pm u)}$. Show that $z^2 = w$ if $v \geq 0$. And $(\overline{z})^2 = w$ if $v \leq 0$.

3.6. Bonus Problem

(a) Fix $b \in \mathbb{R}$ with $b \geq 1$. For $x \in \mathbb{R}$ define $P(b,x) = \{b^t | t \in \mathbb{Q}, t \leq x\}$. Prove that :

$$b^r = P(b,r) \text{ for all } r \in \mathbb{Q}$$

Hence it makes sense to define :

$$b^x = \sup_{x \in \mathbb{R}} P(b,x)$$

(b) Prove that $b^{x+y} = b^x b^y$ for all $x, y, b \in \mathbb{R}$ with $b \geq 1$.