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Analysis I —Assignement 4

4.1. *Squeezing theorem*

Let $(a_n), (b_n)$ and (c_n) be three sequences of real numbers such that : $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = b \in \mathbb{R}$ in $(\mathbb{R}, d)$ where d is the standard metric on $\mathbb{R}$. Show that $\lim_{n\to\infty} b_n = b$ in $(\mathbb{R}, d)$ as well.

4.2. *Fibonacci sequence*

Consider the sequence of real numbers which is inductively defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for all $n \in \mathbb{N}$

(a) Does the sequence (a_n) converge in $(\mathbb{R}, d)$?

(b) Show that the sequence defined by $x_n = \frac{a_{n+1}}{a_n}$ for all $n \in \mathbb{N}$, converges in $(\mathbb{R}, d)$ and determine its limit.

4.3. *Products of sequences*

Let $(a_n), (b_n) \subset \mathbb{R}$ be sequences of real numbers so that :

$$\lim_{n\to\infty} a_n = +\infty, \lim_{n\to\infty} b_n = 0.$$

The limit of the sequence of the products $(a_n b_n)$ can realize very different values depending on the sequences a_n, b_n. Find one example for each of the following cases :

(a) $\lim_{n\to\infty} (a_n b_n) = +\infty.$

(b) $\lim_{n\to\infty} (a_n b_n) = -\infty.$

(c) $\lim_{n\to\infty} (a_n b_n) = 0.$

(d) $\lim_{n\to\infty} (a_n b_n) = c$ where $c \in \mathbb{R}$.

(e) $(a_n b_n)$ is bounded but does not converge.

(f) $(a_n b_n)$ is unbounded but does not converge to either $+\infty$ nor $-\infty$.

4.4. *A real valued sequence* Discuss the convergence of the real valued sequence given by :

$$a_1 = 1, \ a_{n+1} = \frac{2(2a_n + 1)}{a_n + 3} \text{ for } n \in \mathbb{N}.$$

4.5. *Some simple limits* Determine the convergence or divergence of the following sequences in $\mathbb{R}$:

(a) $a_n = \frac{3n}{n+1}$

(b) $b_n = \sqrt[3]{n^3 + n^2} - \sqrt[3]{n^3 - n^2}$

(c) $c_n = n(\sqrt[n]{n} - 1)$

4.6. *Lim sup and Lim inf properties* Given sequences (a_n) and (b_n) in $\mathbb{R}$.

(a) Show that $\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty}(a_n + b_n) \leq \liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$

(b) If the limit of (a_n) exists, prove that $\limsup_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$

(c) Show that $\limsup_{n\to\infty} b_n = \lim_{N\to\infty} (\sup\{b_n : n \geq N\})$ and give a similar characterization for $\liminf_{n\to\infty} b_n$

4.7. *Distance from a point to a set*

The distance from a point x in a metric space (X, d) to a nonempty subset $S \subseteq X$ is defined to be :

$$dist(x, S) = \inf\{d(x, s) : s \in S\}.$$

Show that there is a sequence $(s_n)_n \subset S$ with $\lim_{n\to\infty} s_n = x$, if and only if $dist(x, S) = 0$

4.8. *Metrics in* $\mathbb{R}^n$

(a) Show that the map $d_\infty : \mathbb{R}^n \to \mathbb{R}^n$ $(x, y) \to := \sup\{\|x_i - y_i\| : i = 1, \cdots, n\}$ is a metric on $\mathbb{R}^n$.

(b) Let $X = \{[a, b] \subset \mathbb{R} : a < b\}$ and $Y = \{(a, b) \subset \mathbb{R} : a < b\}$ and

$$d([a, b], [c, d]) := \inf\{\epsilon > 0 : [a, b] \subset [c - \epsilon, d + \epsilon] \text{ and } [c, d] \subset [a - \epsilon, b + \epsilon]\}$$

Are (X, d) and/or $(X \cup Y, \overline{d})$ metric spaces, where $\overline{d}$ is the canonical extension od d onto $X \cup Y$?

4.9. Bonus Problem

(a) Suppose $0 < b < a$. Define: $x_0 = a, y_0 = b$

$$x_{n+1} = \frac{x_n + y_n}{2}, \; y_{n+1} = \frac{2x_n y_n}{x_n + y_n}$$

Show that the sequences have the same limit and find that limit.

(b) Now suppose $0 < a < b$ and let : $x_0 = a, y_0 = b$

$$x_{n+1} = \frac{x_n + y_n}{2}, \; y_{n+1} = \sqrt{x_{n+1} y_n}$$

Show that the sequences have the same limit and find that limit.