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Analysis I —Assignement 5

5.1. *Real sequence*

Prove that the sequence $x_n = (n+1)^p - n^p$ has a limit for every $p \in \mathbb{R}$. Call this limit L_p and discuss the values of L_p as p varies.

5.2. *Some usefull inequality* Prove the following inequality for $p \in \mathbb{R}, p \geq 1$

$$(a_1^p + a_2^p + \cdots + a_n^p)^{\frac{1}{p}} + (b_1^p + b_2^p + \cdots + b_n^p)^{\frac{1}{p}} \geq (a_1 + b_1)^p + (a_2 + b_2)^p + \cdots + (a_n + b_n)^p)^{\frac{1}{p}}$$

where $(a_k)_k, (b_k)_k$ are nonnegative real sequences. What is the interpretation of this inequality ? (Think of $\mathbb{R}^n$)

5.3. *Absolute convergence* Let $(z_n)_n$ be a sequence of complex numbers such that the sequence $(A_n)_n$ given by $A_n = \|z_1\| + \|z_2\| + \cdots + \|z_n\|$ is convergent, where $\|a + ib\| = \sqrt{a^2 + b^2}$. Show that the sequence $B_n = z_1 + z_2 + \cdots + z_n$ is also convergent. Does it follow that if B_n is convergent then A_n is convergent.

Hint: You should think of Cauchy sequences and their properties

5.4. *Density and Jacobi's theorem*

Given a metric space (X, d) and a subset $Y \subset X$, we say that Y is dense in X if for every $x \in X$ there is a sequence $(y_n)_n \subset Y$ such that $\lim_n y_n = x$. Is $\mathbb{Q}$ dense in $\mathbb{R}$? How about $\mathbb{R}$ in $\mathbb{C}$. Argue why. Now let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $M = \{m + n\alpha : m, n \in \mathbb{Z}\}$. Show that M is dense in $\mathbb{R}$.

5.5. *Stolz-Caesaro lemma*

Let $(a_n)_n, (b_n)_n$ be two real sequences such that $(b_n)_n$ is strictly increasing and unbounded. Prove that :

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

given that the limit on the right exists. What if the left limit exists, can you say anything about the right limit?

Hint:Try to use $\limsup$ and $\liminf$

5.6. Bonus Problem

Let (X, d) be a metric space. Prove the following statements :

(a) If $(x_n)_n, (y_n)_n$ are two Cauchy sequences in X then $(d(x_n, y_n))_n$ converges in $\mathbb{R}$.

(b) For two sequences $(x_n)_n, (y_n)_n$ in X we say that $(x_n) \sim (y_n)$ iff $\lim_n d(x_n, y_n) = 0$. Prove that " $\sim$ " is an equivalence relation.

(c) For $[x_n], [y_n] \in X/\sim$ define $\widetilde{d}([x_n], [y_n]) = \lim_n d(x_n, y_n)$. Then $\widetilde{d}$ is well-defined and $(X/\sim, \widetilde{d})$ is a complete metric space. (Notice that you have to prove three things here)