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Analysis I — Problem Set 6

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6.1. Cauchy condesation test Let $(a_n)_n$ be a decreasing sequence. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the series $\sum_{n=1}^{\infty} 2^n a_{2^n}$ is convergent.

6.2. Radius of convergence. Find the radius of convergence of each of the following power series:

$$\text{(a)} \sum_{n=0}^{\infty} \frac{2^n}{n!} z^n, \qquad \text{(b)} \sum_{n=0}^{\infty} \frac{2^n}{n^2} z^n, \qquad \text{(c)} \sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n.$$

You may use the fact that $\lim_{n\to\infty} \sqrt[n]{n} = 1$

6.3. Series and sequences.

(a) Determine whether the series $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$ is convergent or not.

(b) Find the limit of the sequence $a_n := \left(1 - \frac{1}{n}\right)^n$.

6.4. Leibniz criterion (Alternating series test) Let (a_n) be a monotonically decreasing sequence with $a_n > 0$ and $\lim_{n\to\infty} a_n = 0$. The the series$\sum_{n=0}^{\infty}(-1)^n a_n$ converges.

6.5. Approximating the harmonic series Let (a_n) be a sequence in $(\mathbb{R}, d_1)$ such that $a_n > 0$ for all $n \in \mathbb{N}$. Show that if $\lim_{n\to\infty}(na_n) = l$ with $l \neq 0$, then $\sum a_n$ diverges.

6.6. Converging series Determine the limit of the following sequences:

(a) $\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{(2n+1)^2}$

(b) $\lim_{N\to\infty} \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^2}$

You may use the fact that $\sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}$.

6.7. Approximation of Real Numbers. Show that for every real number $x \in (0,1)$ there are integers $1 < n_1 < n_2 < \ldots$ such that

$$x = \sum_{k=1}^{\infty} \frac{1}{n_k}$$

Hint : Think of the harmonic series

6.8. Abel's criterion Let $(a_n)_n, (b_n)_n$ be two sequences such that the series $A_N = \sum_{n=1}^{N} a_n$ is convergent and $(b_n)_n$ is decreasing and bounded. Then the series $S_N = \sum_{n=1}^{N} a_n b_n$ is also convergent.

6.9. Even more series Determine which of the following series converge or not :

(a) $\sum_{n=1}^{\infty} \frac{1}{n^p}, p > 0$ (discussion on p)

(b) $\sum_{n=1}^{\infty} \frac{1}{n \log n}$

(c) $\sum_{n=1}^{\infty} \cos n$

(d) $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$

(e) $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin n$

6.10. BONUS PROBLEM Let $(a_n)_n$ be a real sequence such that $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} |a_n| = \infty$. Prove that for every real number $r \in' R$ there exist a bijective function $f : \mathbb{N} \to \mathbb{N}$ (permutation of the naturals) such that $\sum_{n=1}^{\infty} a_f(n) = r$. Is this true for a complex sequence?