International University Bremen
School of Engineering and Science
Götz Pfander, Alexandru Sava

Fall Term 2010

Analysis I —Assignement 7

7.1. *Discrete metric spaces* Let (X, d_0) be a metric space with $d_0(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$.

(a) Prove that every subset of X is clopen.

(b) Prove that every function defined on X is continuous.

(c) Which sequences converge in X?

7.2. *Metric spaces*

Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. Prove that the following are equivalent:

(a) The function f is continuous.

(b) For all $x_0 \in X$ and for all sequences $(x_n)_{n\in\mathbb{N}}$ in X with $\lim\limits_{n\to\infty} x_n = x_0$ we have $\lim\limits_{n\to\infty} f(x_n) = f(x_0)$.

7.3. *Continuous functions*

Check using the $\varepsilon - \delta$ definition if the following functions are continuous on the real line: $f(x) = [x]$ where x represents the biggest integer $\leq x$; $f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ a, & x = 1 \end{cases}$;

$p(x) = a_n x^n + \dots + a_1 x + a_0$ with $a_k \in \mathbb{R}, k \in \overline{0 \dots n}$; $\dfrac{2x}{1+x^2}$

7.4. *Properties of continuous functions*

Prove the following properties of continuous functions $f : X \to Y$ where $(X, d_1), (Y, d_2)$ are metric spaces:

(a) If $K \subset Y$ is closed then $f^{-1}(K)$ is also closed. Is it always true that if $K \subset X$ is closed then $f(K)$ is also closed ? Prove or give counterexamples.

(b) If $K \subset X$ is connected then $f(K)$ is connected. (A subset $E \subset X$ is called connected if for every 2 open subsets $A, B \subset X$ with $A \cap B = \emptyset$ and $X \subset (A \cup B)$ then either $X \subset A$ or $X \subset B$)

(c) For a subset $E \subset X$ define its closure $\overline{E}$ by the intersection of all closed sets in X that contain E. Then $\overline{E}$ is a closed set for any E and $f(\overline{E}) \subset \overline{f(E)}$. Find an example where the inclusion is proper.

7.5. *A special function*

Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows: $f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \dfrac{1}{n} & \text{if } x = \dfrac{m}{n} \text{ in lowest terms, } n > 0 \end{cases}$

Prove that f is continuous on the set of irrational numbers and discontinuous on the rationals.

7.6. *Convex implies continuous*

A function $f : \mathbb{R} \to \mathbb{R}$ is called convex if for every $\alpha, \beta > 0, \alpha + \beta = 1$ we have that :

$$\alpha f(x) + \beta f(y) \geq f(\alpha x + \beta y)$$

for any $x, y \in \mathbb{R}$. Prove that every convex function is also continuous.

7.7. *Fixed point*

Let $f : [0, 1] \to [0, 1]$ continuous. Prove that there exist $c \in [0, 1]$ such that $f(c) = c$.

7.8. *Rudin page 98 problem 3 and more*

Let f be a continuous real valued function on a metric space X. Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ at which $f(p) = 0$.

(a) Prove that $Z(f)$ is closed.

(b) Recall that for a set $E \subset X$ the distance from a point to this set is defined as

$$h(x) = \inf_{s \in E} d(x, s)$$

Prove that h is uniformly continuous

(c) Use the previous part to show that for any closed set $E \subset X$ there exist a continuous function $f : X \to \mathbb{R}$ that is 0 on E and positive elsewhere.

7.9. Bonus Problem

Let $f : \mathbb{R} \to \mathbb{R}$. For a discontinuity point c we say c is of type 1 if $\lim_{x \to c, x < c} f(x) = \lim_{x \to c, x > c} f(x) \neq f(c)$ or $\lim_{x \to c, x < c} f(x) \neq \lim_{x \to c, x > c} f(x)$. Call c of type 2 if at least one of the previous limits doesn't exist or it is infinite.

(a) Give examples of functions to illustrate the existence of all types of discontinuities.

(b) Show that if f is monotone then f has only discontinuities of type 1 and there are countable many.