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Analysis I —Assignement 8

8.1. *Properties of compact sets*

Let (X, d) be a metric space. Prove that :

(a) If $K \subset X$ is compact then every $K' \subset K$ which is closed in K is also compact in X.

(b) Finite unions of compact sets are compact.

(c) Arbitrary intersections of compact sets are compact.

8.2. *A nice topological property*

Let (X, d) be a metric space. Prove that :

(a) For every $x, y \in X, x \neq y$ there exist $r, r' > 0$ such that $B_r(x) \cap B_{r'}(y) = \emptyset$ (Hausdorff property)

(b) If $x \in X$ then x is a closed set.

(c) Every compact set in X is closed.

8.3. *Compact sets on the real line*

Consider $\mathbb{R}$ with the standard metric. Show that:

(a) If $K \subset \mathbb{R}$ is compact then it is bounded.
Hint: Consider coverings by intervals of length 1.

(b) The closed interval $[0, 1]$ is compact.
Hint: First show a covering with arbitrary open sets implies a covering with open intervals. Through out the redundant intervals. How many are there ? (Midterm problem) Pick out "relevant" points from the remaining intervals. They must accumulate around a point p. Why? Where does p live?

(c) If $K \subset \mathbb{R}$ is closed and bounded then it is compact. Conclude that compactness in $\mathbb{R}$ is equivalent with begin closed and bounded.
Hint: If K is bounded then it lies inside a closed interval

(d) $K \subset \mathbb{R}$ is compact iff every sequence $(x_n)_n \subset K$ has a convergent subsequence in K.

NOTE: This problem is worth 20 points instead of the usual 10 since it's more involved.

8.4. *Property of continuous functions*

Let $(X, d), (Y, d')$ be metric spaces such that X is compact. Prove that every continuous function $f : X \to Y$ is bounded.

8.5. *The closure, boundary and interior of a set*

Let A be a set in a metric space X. Recall that A' is the set of cluster points of A, $\overline{A}$ is the closure of A, A° is the interior of A, and ∂A is the boundary of A. Prove or give counterexamples to the following statements:

(a) $A \subseteq \partial A,\ \partial A \subseteq A$.

(b) $A \subseteq \overline{A},\ \overline{A} \subseteq A$.

(c) $A \subseteq A^\circ,\ A^\circ \subseteq A$.

(d) $A \subseteq A',\ A' \subseteq A$.

(e) $\partial A \subseteq \partial\partial A,\ \partial\partial A \subseteq \partial A$.

(f) $\overline{A} \subseteq \overline{\overline{A}},\ \overline{\overline{A}} \subseteq \overline{A}$.

(g) $A^\circ \subseteq A^{\circ\circ},\ A^{\circ\circ} \subseteq A^\circ$.

(h) $A' \subseteq A'',\ A'' \subseteq A'$.

(i) $A' \subseteq \overline{A},\ \overline{A} \subseteq A'$.

(j) $A \cup A' \subseteq \overline{A},\ \overline{A} \subseteq A \cup A'$.

8.6. *Cnt's fctn's are determined by image on a dense subset, Rudin page 98 problem 4*

Let $f, g : X \to Y$ be continuous maps between two metric spaces. Let $E \subset X$ be a dense subset. Prove that $f(E)$ is dense in $f(X)$. If $f(x) = g(x)$ for all $x \in E$ then $f = g$. (In other words, a continuous map is completely determined by the image on a dense subset of its domain)

8.7. **BONUS PROBLEM A nice characterization of continuity, Rudin page 99 problem 6** If f is defined on E, the graph of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers then its graph is a subset of the plane. Suppose E is a compact metric space. Show that a function f defined on E is continuous iff its graph is compact.