International University Bremen
School of Engineering and Science
Götz Pfander, Alexandru Sava

Fall Term 2010

Analysis I —Assignement 9

9.1. *Continuous functions on compact sets*

Let $(X, d), (Y, d')$ be metric spaces. Prove that :

(a) If $K \subset X$ is compact then every continuous function $f : X \to Y$ is uniformly continuous on K i.e. for every $\varepsilon > 0$ there exist $\delta > 0$ such that for every $x, y \in K$ with $d(x, y) < \delta$ we have $d'(f(x), f(y)) < \varepsilon$.

(b) If X is compact and $f : X \to Y$ is continuous and bijective then f is a homeomorphism i.e. f is continuous and bijective with a continuous inverse.

(c) Give an example of metric spaces X, Y and a continuous, bijective function $f : X \to Y$ such that f^{-1} is not continuous. Explain your example.

9.2. *Perfect sets*

A set is called "perfect" if it is closed and doesn't contain any isolated points.

(a) Show that if a subset A of a metric space X is perfect then $A = A'$. Is the converse true? Explain !

(b) Show that a nonempty perfect set A in a metric space $\mathbb{R}$ with the usual metric is uncountable.

Comment: This is true for more general metric spaces called Polish spaces. These are metric spaces which are complete and have a countable dense subset. $\mathbb{R}$ with the usual metric is such a space. Try to argue this statement in full generality.

(c) Recall the construction of the Cantor set: start with the interval $C_0 = [0, 1]$. At step n you have a set C_n which consists of 2^n intervals of size 3^{-n}. C_{n+1} is obtained by picking each subinterval in C_n and cutting the middle open interval of size $3^{-(n+1)}$. So C_{n+1} will consists of 2^{n+1} intervals of size $3^{-(n+1)}$. For instance $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Show that $C_0 \supset C_1 \supset C_2 \supset \cdots$. Define $C = \bigcap_n C_n$ the standard Cantor set. Show that C is a compact, nonempty perfect set. (Hint: it is bounded so it is enough to show it is perfect. Why?)

(d) Prove that C defined above is totally disconnected i.e it contains no intervals.

NOTE: This problem is worth 20 points !

9.3. *Composition of continuous and uniformly continuous*

Suppose X, Y, Z are metric spaces and Y is compact. Let f map X into Y, let g be a continuous one-to-one (injective) mapping of Y into Z, and put $h(x) = g(f(x))$ for $x \in X$. Prove that:

(a) f is continuous if h is continuous.

(b) If h is uniformly continuous then f is uniformly continuous. Is the converse true ? i.e. if f is uniformly continuous then h is uniformly continuous? What if we also assume the g is uniformly continuous?

9.4. *Uniform convergence of power series.*

Let $D = \{z \in \mathbb{C},\, |z| \leq 1\}$ be the unit ball in the complex plane and $\mathrm{X} = \{f : D \to \mathbb{C},\, f \text{ is bounded}\}$ and $d_u(f, g) = \sup\{|f(z) - g(z)|,\, z \in D\}$.

(a) Show that (X, d_u) is a metric space.

(b) Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 1$. Show that $f_{|D} \in \mathrm{X}$ and $f_{N|D} \in \mathrm{X}$, where $f_N : D \to \mathbb{C}, z \mapsto \sum_{n=0}^{N} a_n z^n$.

NOTE : $g_{|D}$ is the restriction of g on D. In other words if the domain of g is E such that $D \subseteq E$ then $g_{|D}$ is a function defined on D such that $g_{|D}(x) = g(x)$ for all $x \in D$.

(c) Prove that $\lim_{N \to \infty} f_N = f$ in the metric space (X, d_u).

Note: Recall that $f : A \to \mathbb{C}$ is bounded if there exists $M \in \mathbb{R}$ such that $|f(a)| \leq M$ for all $a \in A$.

9.5. *Uniform metric*

Consider again $\mathrm{X} = \{f : D \to \mathbb{C},\, f \text{ is bounded}\}$ with metric $d_u(f, g) = \sup\{|f(z) - g(z)|,\, z \in D\}$. Show that $\{f \in \mathrm{X} :\ f(0) = 0\}$ is closed in (X, d_u).

9.6. BONUS PROBLEM Cantor-Bendixson Theorem

Let X be a Polish space (a complete metric space with a countable dense subset). Then any closed set $C \subseteq X$ may be written uniqueley as the disjoint union between a perfect set P and a countable set S. That is $C = P \cup S$ with $P \cap S = \emptyset$.