International University Bremen
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Spring Term 2008
Thursday, May 22th
15:00 – 17:00

Analysis II — Final Exam

Notes: Sign your work to certify that you adhere to the academic Code of Honor to work independently. You may use and cite all results within the script, the homeworks, and the examinations. *All answers must be justified! Show all your work!*

Each problem is worth 60 points, do any 5 of them. Points achieved beyond 300 points are bonus points.

F1. (a) Compute $\displaystyle\int \frac{\sqrt{t^2+1}}{t}\,dt.$

(b) Find the length of the curve $X(t) = (t, \log t)$ between $t = 1$ and $t = 2$.

F2. Let f be defined on the square $S = [0,1] \times [0,1]$ in the following way:

$$f(x,y) = \begin{cases} 1 & \text{if } x \text{ is irrational,} \\ 4y^3 & \text{if } x \text{ is rational.} \end{cases}$$

Compute the following integrals or argue that they do not exist:

a) $\displaystyle\int_0^1 \int_0^1 f(x,y)\,dydx,$ b) $\displaystyle\int_0^1 \int_0^1 f(x,y)\,dxdy,$ c) $\displaystyle\int_S f.$

F3. Compute the double integral $\displaystyle\int_D (|x| + |y|)\,dxdy$, where $D \subset \mathbb{R}^2$ is the ellipse, defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1,$$

where a, $b > 0$.

F4. Let $f(x,y,z) = x^3 - 2y^2 + z^2$. Show that $f(x,y,z) = 0$ defines an implicit function $x = \phi(y,z)$ at the point $(1,1,1)$. Find $\operatorname{grad}\phi$ at the point $(1,1)$.

F5. Let $I \subset \mathbb{R}$ be an open interval and assume that $f : I \times I \to \mathbb{R}$ has continuous partial derivatives. Let the function $F : I \longrightarrow \mathbb{R}$ be defined by

$$F(y) := \int_a^y f(x,y)\,dx.$$

Prove that the function F is differentiable and

$$F'(y) = f(y,y) + \int_a^y \frac{\partial f(x,y)}{\partial y}\,dx.$$

[Hint: Consider first $G(z,y) := \displaystyle\int_a^z f(x,y)\,dx$.]

F6. (a) Find the solution of the following differential equation satisfying the initial condition $y(1) = 1$:

$$xy' - 2y = 2x^4,$$

(b) Find the general solution of the following homogeneous differential equation:

$$y^2 + x^2y' = xyy'.$$

[Note that the solution may be implicit.]