Jacobs University Bremen
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Spring Term 2008
Tuesday, April 1^{st}
11:15 – 12:30

Analysis II — Midterm Exam

Notes: Sign your work to certify that you adhere to the academic Code of Honor to work independently. You may use and cite all results within the script, the homeworks, and the examinations. *All answers must be justified! Show all your work!*

Each problem is worth 44 points. We count your best five solutions.

M1. Integration. Show that

$$\int_0^\infty \frac{\cos x}{1+x}\,dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx.$$

M2. Integration.

(a) Show that for $b > a > 0$ we have

$$\int_a^b \frac{1}{1+x^2}\,dx = \int_{1/b}^{1/a} \frac{1}{1+x^2}\,dx. \tag{1}$$

(b) Does (1) also hold for $b > 0 > a$?

M3. Fourier series. Let $g : \mathbb{R} \to \mathbb{C}$ be 1-periodic and continuous and let $h_a : \mathbb{R} \to \mathbb{C}$ be given by $h_a(x) = g(x-a)$, $a \in \mathbb{R}$. Show that $\widehat{h_a}(n) = \widehat{g}(n)e^{2\pi ian}$ for $n \in \mathbb{Z}$.

M4. Calculus. Show that if f is a continuously differentiable real-valued function on $U = (-1,1) \times (-1,1)$ and $\dfrac{\partial^2 f}{\partial x \partial y}$ exists on U with $\dfrac{\partial^2 f}{\partial x \partial y} = 0$ on U, then there are continuously differentiable real-valued functions f_1, f_2 on $(-1,1)$ such that $f(x,y) = f_1(x) + f_2(y)$ for $(x,y) \in U$.

M5. Multivariable calculus. Let $f, F : \mathbb{R}^3_+ \to \mathbb{R}$ are differentiable functions, such that $f(x,y,z) = F(u,v,w)$ whenever $x^2 = vw$, $y^2 = uw$, $z^2 = uv$. Prove that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w}.$$

Hint: Apply the chain rule with, for example, $x(u,v,w) = \sqrt{vw}$.

M6. Multivariable calculus. Let f be a real-valued function of class C^2 on some open ball U in $\mathbb{R}^n$ centered in x_0. Show that

$$f(x) = f(x_0) + (Df)_{x_0}(x - x_0) + g(x)(x - x_0, x - x_0),$$

where $g : U \to \mathcal{L}^2(\mathbb{R}^n, \mathbb{R})$ is a continuous map.

Hint: Use a mean value theorem.