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Analysis II — Problem Set 11

Issued: 29.04.08 Due: 07.05.08

11.1. Show that if $\varphi : D_t \longrightarrow D_x$, $D_t, D_x \subseteq \mathbb{R}^n$ open and bounded, is a diffeomorphism, then $\varphi(E_t)$ is a zeroset whenever $E_t \subseteq D_t$ is a zeroset.

11.2. Consider the following sequence of integrals:

$$F_0(x) = \int_0^x f(y)dy, \; F_n(x) = \int_0^x \frac{(x-y)^n}{n!} f(y)dy, \; n \in \mathbb{N},$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

(a) First check that $F_n'(x) = F_{n-1}(x)$; $F_n^{(k)}(0) = 0$, if $k \leq n$; $F_n^{n+1}(x) = f(x)$.

(b) Show that

$$\int_0^x dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{n+1}} f(x_n)dx_n = \frac{1}{n!} \int_0^x (x-y)^n f(y)dy.$$

11.3. (a) Does the following integral exist?

$$\lim_{A \longrightarrow \infty} \int_0^A \cos x^2 dx.$$

(b) Check for convergence of

$$\lim_{n \longrightarrow \infty} \int_{-n}^n \cos x^2 dx$$

(c) Check that

$$\lim_{n \longrightarrow \infty} \int\limits_{|x| \leq n} \sin(x^2 + y^2) d(x, y) = \pi$$

and

$$\lim_{n \longrightarrow \infty} \int\limits_{x^2 + y^2 \leq 2\pi n} \sin(x^2 + y^2) d(x, y) = 0$$

and conclude that the improper integral $\int_{R^2} \sin(x^2 + y^2) d(x, y)$ does not exist.

11.4. By considering both orders of integration in

$$\int_0^X \int_a^b e^{-tx} dt dx,$$

where $b > a > 0$ and $X > 0$, show that

$$\int_0^X \frac{e^{-ax} - e^{-bx}}{x} dx = \int_a^b \frac{1 - e^{-tX}}{t} dt.$$

Show that

$$\int_a^b \frac{e^{-tX}}{t} dt \leq e^{-aX} \log\left(\frac{a}{b}\right),$$

and hence find

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx.$$

11.5. Compute the following integral

$$\int_{x^2+y^2>1} \frac{|xy|}{x^2+y^2} e^{-x^2-y^2} dxdy.$$

Hint: use polar coordinates.

11.6. (Bonus problem)

(a) Using the Beta function

$$B(x,y) = \int_0^\infty t^{x-1}(1-t)^{y-1} dt, \tag{1}$$

prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
Hint: use the substitution $t = \sin^2\theta$ in (1) to compute $B(\frac{1}{2}, \frac{1}{2})$.

(b) Compute the integral

$$\int_{-\infty}^\infty e^{-s^2} ds.$$

Hint: use the substitution $t = s^2$ in the definition of the Gamma function.