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Analysis II — Problem Set 2

Issued: 12.02.08 Due: 20.02.08, noon

2.1. Natural Logarithm
Compute the following integral using Riemann sums

$$\int_1^a \frac{dx}{x}, \text{ where } a > 1.$$

Hint: Consider a partition $1 = x_0 < x_1 < \ldots < x_n = a$, where $x_k := a^{k/n}$ and take $\xi_k := x_{k-1}$ as midpoints.

2.2. Riemann integrability and composition
Show that if $f : [a,b] \longrightarrow [c,d]$ is Riemann integrable and $\phi : [c,d] \longrightarrow \mathbb{R}$ is continuous, then the composite $\varphi \circ f$ is Riemann integrable.

2.3. Total variation
Given a function f on $[a,b]$, define the *total variation* of f to be

$$Vf = \sup\left\{\sum_{k=1}^{n} \left|f(x_{k-1}) - f(x_k)\right|\right\},$$

where the supremum is taken over all partitions $P = \{x_0, x_1, \ldots x_n\}$ of $[a,b]$.

(a) If f is continuously differentiable, use the Fundamental Theorem of Calculus to show $Vf \leq \int_a^b |f'|$.

(b) Use MVT to establish the reverse inequality and conclude that $Vf = \int_a^b |f'|$.

2.4. Let

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} .$$

Prove that f is integrable on $[-1,1]$ and $F(x) = \displaystyle\int_{-1}^{x} f(t)dt$ is differentiable on $(-1,1)$ and find $F'(0)$.

2.5. Let $f \in \mathcal{R}[a, b]$. Prove that $\int_a^b f^2(x)dx = 0$ if and only if $f(x) = 0$ at all point $x \in [a, b]$, at which f is continuous.

2.6. Bonus problem.
Let $f \in \mathcal{R}[a, b]$. Prove that there exists a sequence of functions φ_n continuous on $[a, b]$ s.t. $\displaystyle\int_a^b |\varphi_n(x) - f(x)| \longrightarrow 0$ when $n \longrightarrow \infty$.