Jacobs University Bremen
School of Engineering and Science
Götz Pfander, Sergei Markouski,
Alex Sava

Spring Term 2008

Analysis II — Problem Set 4

Issued: 26.02.08 Due: 05.03.08, noon

4.1. (a) Show that $\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2}$ for $x \in (0, 2\pi)$.

Hint: Use 5.14 and 5.39 in the script posted online.

(b) Use (a) to show that $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \left(\frac{x-\pi}{2}\right)^2 - \frac{\pi^2}{12}$, $x \in [0, 2\pi]$, and in particular $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Hint: Use 4.29 for $[a, b] \subset (0, 2\pi)$.

4.2. Fourier series. Prove the following statements:

(a) If $f \in C([0,1])$ with $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f(x) = 0$ for all $x \in [0, 1]$.

(b) We showed that for $f \in \mathcal{R}([0,1])$ we have $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 = \int_0^1 |f(x)|^2 \, dx$. Give a counterexample to $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| = \int_0^1 |f(x)| \, dx$. Check for your counterexample that $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 = \int_0^1 |f(x)|^2 \, dx$ is correct.

(c) If $f \in C([0,1])$ with $\sum_{n=-\infty}^{+\infty} |\widehat{f}(n)|$ convergent, then $S(f, N) \longrightarrow f$ uniformly (and therefore pointwise).

(d) If $f \in \mathcal{R}([0,1])$ with $\sum_{n=-\infty}^{+\infty} |\widehat{f}(n)|$ convergent, then $f(0) = f(1)$.

(e) If $f \in C([0,1])$ with $\sum_{n=-\infty}^{+\infty} |n\widehat{f}(n)|$ convergent, then $f \in C^1([0,1])$ and $f(t) = 2\pi i \sum_{n=-\infty}^{\infty} n\,\widehat{f}(n) e^{2\pi i n t}$.

4.3. Partial derivatives.

(a) Consider the function

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R},\ (x, y) \mapsto \begin{cases} \frac{xy}{x^2+y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$$

and the induced functions

$$F_{y_0} : \mathbb{R} \longrightarrow \mathbb{R},\ x \mapsto \frac{xy_0}{x^2 + y_0^2}, \quad F_{x_0} : \mathbb{R} \longrightarrow \mathbb{R},\ y \mapsto \frac{x_0 y}{x_0^2 + y^2},\ x_0, y_0 \in \mathbb{R}.$$

- Show that F_{x_0}, $x_0 \in \mathbb{R}$, and F_{y_0}, $y_0 \in \mathbb{R}$, are differentiable on $\mathbb{R}$.
- Show that F is not continuous at $(0, 0)$ (consequently, it is not differentiable at $(0, 0)$).

(b) Show that the function

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R},\ (x, y) \mapsto \begin{cases} (x^2 + y^2) \cos \frac{\pi}{x^2+y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$$

is differentiable at $(0, 0)$, but its partial derivatives are discontinuous at $(0, 0)$.