Jacobs University Bremen
School of Engineering and Science
Götz Pfander, Sergei Markouski,
Alex Sava

Spring Term 2008

Analysis II — Problem Set 5

Issued: 04.03.08 Due: 12.03.08

5.1. (a) If f is a real function defined on a convex open set $U \subset \mathbb{R}^n$ such that the partial $D_1 f = 0$ for every $x \in U$, show that $f(x)$ depends only on $x_2, x_3, \ldots, x_n$.

(b) Given an example of a function f and a non convex open set U where $D_1 f = 0$ for every $x \in U$ but $f(x)$ depends on x_1.

5.2. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}^3$ is a differentiable mapping with $\|f(x)\| = 1$ for all $x \in \mathbb{R}$. Show that $\langle f(x), f'(x)\rangle = 0$ for all $x \in \mathbb{R}$ and give a geometric interpretation.

5.3. Provide Matlab or Mathematica images of your solutions to the following problems.

(a) Recall that for

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R},\ (x, y) \mapsto \begin{cases} \frac{xy}{x^2+y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$$

both partials exist. Construct a differentiable function g such that $f \circ g$ fails to have a partial.

(b) Similarly, for

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R},\ (x, y) \mapsto \begin{cases} \frac{x^3 y}{x^4+y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$$

all directional derivatives exist. Construct a differentiable function g such that $f \circ g$ fails to have some directional derivatives.

5.4. Let $f : U \longrightarrow \mathbb{R}^m$, $U \subset \mathbb{R}^n$ open be differentiable, and $[p, q] = \{tp + (1 - t)q,\ t \in [0, 1]\} \subset U$. This problem answers the question whether the direct generalization of the one–dimensional Mean Value Theorem is true: does there exist a point $\theta \in [p, q]$ such that

$$f(q) - f(p) = (Df)_\theta (q - p)? \tag{1}$$

(a) Take $n = 1, m = 2$ and examine the function $f(t) = (\cos t, \sin t)$ for $0 \le t \le 3\pi$. For $p = \pi, q = 2\pi$, does a θ exist satisfying (1)?

(b) Assume that the set of derivatives

$$\{(Df)_x \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) : x \in [p, q]\}$$

is convex. Prove that then a θ exists satisfying (1).

5.5. Bonus problem. Prove that for $x \in [0,1]$, we have $\displaystyle\sum_{n=1}^{\infty} \frac{\cos 2\pi n x}{n^2} = \pi^2 \left(\frac{(2x-1)^2}{4} - \frac{1}{12} \right)$.

[*Hint*: Use 5.14 and 5.29 in the script.]

5.6. Bonus problem. Total derivative.

(a) Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(x,y) \mapsto (x^3,\ y^3 - x^2)$. Calculate the differential Df of f and show that for all $(x,y) \in \mathbb{R}^2$

$$f(x+h_x, y+h_y) - f(x,y) = (Df)_{(x,y)}(h_x,h_y) + R_{x,y}(h_x,h_y),$$

where $\frac{||R_{x,y}(h_x,h_y)||}{||(h_x,h_y)||} \longrightarrow 0$ for $(h_x,h_y) \longrightarrow (0,0)$.

(b) Let f be as in (a) and define the function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x,y) \mapsto e^{-(x^2+y^2)}$. Calculate Dg and $Dg \circ f$ and verify for those two functions the chain rule

$$\big(Dg \circ f\big)(x,y) = \big(Dg)(f(x,y))\big)\,\big(Df\big)(x,y).$$

5.7. Bonus problem. Function $f(x,y)$ is defined and continuous in some neighborhood of point (x_0,y_0) and satisfies the following conditions:

(a) Partial derivatives $f'_x(x_0,y_0)$ and $f'_y(x_0,y_0)$ exist;

(b) For any function $\varphi : \mathbb{R}^2_{uv} \longrightarrow \mathbb{R}^2_{xy}$, $\varphi(u_0,v_0) = (x_0,y_0)$, continuously differentiable in some neighborhood of (u_0,v_0), there exist derivatives $(f(\varphi))'_u(u_0,v_0) = A$ and $(f(\varphi))'_v(u_0,v_0) = B$;

(c) the following equalities hold

$$A = f'_x(x_0,y_0)x'_u(u_0,v_0) + f'_y(x_0,y_0)y'_u(u_0,v_0),$$

$$B = f'_x(x_0,y_0)x'_v(u_0,v_0) + f'_y(x_0,y_0)y'_v(u_0,v_0)$$

Prove that f is differentiable at (x_0,y_0).