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Analysis II — Problem Set 6

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6.1. Introducing new independent variables u, v and a new function $\omega(u, v)$, simplify the equation

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - 2z = 0,$$

given that

$$u = \frac{x}{y},\ v = \frac{x^2 + y^2}{2},\ w = \frac{xy}{z}.$$

6.2. We say that $f : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}$ is homogeneous of degree $\alpha > 0$ if

$$f(\lambda x) = \lambda^\alpha f(x) \quad \text{for all } \lambda > 0 \text{ and } x \neq 0. \tag{1}$$

Show that for any homogeneous and differentiable function $f : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}$ we have

$$\sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j}(x) = \alpha f(x).$$

6.3. Assume that $f : [a, b] \times (c, d) \longrightarrow \mathbb{R}$ is continuous and $\frac{\partial f}{\partial y}$ exists and is continuous on $[a, b] \times (c, d)$. Show that then $F(y) = \int_a^b f(x, y)\,dx$ is $C^1(c, d)$, and

$$F'(y) = \int_a^b \frac{\partial f}{\partial y}(x, y)\,dx, \quad y \in (c, d)\,.$$

6.4. (a) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be differentiable with $\|(Df)_x\|_{\mathcal{L}} \leq c < 1$ for all $x \in \mathbb{R}^n$. Show that f has a fixed point.

(b) Apply the above to show that

$$f(x, y) = \begin{pmatrix} \frac{\cos x}{2} \\ \frac{x+y}{4} \end{pmatrix}, \quad (x, y) \in \mathbb{R}^2,$$

has a fixed point.

6.5. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$, be a differentiable function, $v = (v_1, ..., v_n) : U \to \mathbb{R}^n$ be a differentiable vector field.

(a) Prove that $\mathrm{div}(fv) = \langle \mathrm{grad} f, v\rangle + f \mathrm{div} v$.

(b) Find $\mathrm{div} F$ for a vector field $F : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$, $F(x) = \frac{x}{\|x\|}$.

6.6. Let $U \subset \mathbb{R}^3$ be open and $f : U \to \mathbb{R}$, be a twice differentiable function, $v : U \to \mathbb{R}^3$ be a twice differentiable vector field.

(a) Show that rot grad $f = 0$.

(b) Show that div rot $v = 0$.