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Spring Term 2008

Analysis II — Problem Set 7

Issued: 26.03.08 Due: 02.04.08, noon

7.1. Let $\gamma_1(t) = e^{it}$, $\gamma_2(t) = e^{2it}$, and $\gamma_3(t) = e^{2\pi it\sin(\frac{1}{t})}$, $t \in [0, 2\pi]$, be curves in $\mathbb{R}^2 = \mathbb{C}$. Show that all curves have the same range and calculate the length of those curves which are rectifiable.

7.2. (a) Show that the complete elliptic integral given by the improper integral

$$E(k) = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,dt$$

exists for each $k \in [0, 1]$. $E(k)$ is called complete elliptic integral.

(b) Express the arc length of the ellipse

$$f : [0, 2\pi] \to \mathbb{R}^2,\ t \mapsto (a\cos t, b\sin t)$$

using $E(k)$.

7.3. Let U be an open and connected subset of $\mathbb{R}^n$. Describe all functions $f : U \longrightarrow \mathbb{R}^m$ which have a second derivative everywhere on U and $(D^2f)_x = 0$ for all $x \in U$.

7.4. *Taylor's theorem.* Let E be an open subset of $\mathbb{R}^n$ and f is a real-valued function defined on E. We say that $f \in \mathcal{C}^m(E)$, if all partial derivatives of the function f up to the m^{th} order exist and are continuous. Fix $a \in E$, and suppose $x \in \mathbb{R}^n$ is so close to 0 that the points

$$p(t) = a + tx$$

lie in E whenever $0 \leq t \leq 1$. Define

$$h : [0, 1] \longrightarrow \mathbb{R},\ h(t) = f(p(t)).$$

(a) For $1 \leq k \leq m$, show (by repeated application of the chain rule) that

$$h^{(\mathrm{k})}(t) = \sum (D_{i_1\ldots i_k}f)(p(t))x_{i_1}\ldots x_{i_k}.$$

The sum extends over all ordered k-tuples $(i_1, \ldots, i_k)$ in which each i_j is one of the integers $1, \ldots, n$.

(b) By Taylors's theorem,

$$h(1) = \sum_{k=0}^{m-1} \frac{h^k(0)}{k!} + \frac{h^m(t)}{m!}$$

for some $t \in (0, 1)$. Use this to prove Taylor's theorem in n variables by showing that the formula

$$f(a + x) = \sum_{k=0}^{m-1} \frac{1}{k!} \sum (D_{i_1\ldots i_k}f)(a)x_{i_1}\ldots x_{i_k} + r(x)$$

represents$f(a+x)$ as the sum of its so-called "Taylor polynomial of degree m-1", plus a remainder that satisfies

$$\lim_{x\to 0} \frac{r(x)}{\|x\|^{m-1}} = 0.$$

Each of the inner sums extends over all ordered k-tuples $(i_1, \ldots, i_k)$ as in part (1); as usual, the zero-order derivative of f is simply f, so that the constant term of the Taylor polynomial of f at a is $f(a)$.

(c) Assume that even for higher derivatives, the order of partial derivatives up to degree m does not matter. For example, D_{113} occurs three times, as D_{113}, D_{131}, D_{311}. The sum of the corresponding three terms can be written in the form

$$3(D_1^2 D_3 f)(a) x_1^2 x_3.$$

Prove (by calculating how often each derivative occurs) that the Taylor polynomial in (2) can be written in the form

$$\sum \frac{(D_1^{s_1} ... D_n^{s_n} f)(a)}{s_1! ... s_n!} x_1^{s_1} ... x_n^{s_n}.$$

Here the summation extends over all ordered n-tuples $(s_1, ..., s_n)$ such that each s_i is a nonnegative integer, and $s_i + ... + s_n \leq m - 1$.