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Analysis II — Problem Set 8

Issued: 08.04.08 Due: 16.04.08, noon

8.1. Let f be a real-valued function of class C^2 on some open ball U in $\mathbb{R}^n$ centered in x_0. Show that

$$f(x) = f(x_0) + (Df)_{x_0}(x - x_0) + g(x)(x - x_0, x - x_0),$$

where $g : U \to \mathcal{L}^2(\mathbb{R}^n, \mathbb{R})$ is a continuous map.

8.2. (a) Show that for all $x, y \in \mathbb{R}$, there is exactly one $t \in \mathbb{R}$ such that

$$t^3 + t + xy = 1.$$

Furthermore, show that this defines implicitly a differentiable function $t = g(x, y) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and calculate the gradient of g at the point $(1, 1)$.

(b) Consider the function

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R},\ (x, y_1, y_2) \mapsto x^2 y_1 + e^x + y_2.$$

Show that there is an open neighborhood $U \subset \mathbb{R}^2$ of $(1, -1)$ and a differentiable function $g : U \longrightarrow \mathbb{R}$ such that

$$g(1, -1) = 0 \text{ and } f\,(\,g(y_1, y_2),\, y_1, y_2\,) = 0.$$

Compute the partial derivatives of g at $(1, -1)$.

8.3. Decide whether the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ (x, y) \mapsto (x^2 - y^2,\, 2xy)$ is locally invertible. What about global invertibility?

8.4. Show that the continuity of f' at the point a is needed in the inverse function theorem, even in case $n = 1$: if

$$f(t) = t + 2t^2 \sin\left(\frac{1}{t}\right)$$

for $t \neq 0$, and $f(0) = 0$, then $f'(0) = 1$, f' is bounded in $(-1, 1)$, but f is not one-to-one in any neighborhood of 0.

8.5. Define f in $\mathbb{R}^2$ by

$$f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2.$$

(a) Find the four points in $\mathbb{R}^2$ at which the gradient of f is zero. Show that f has exactly one local maximum and one local minimum in $\mathbb{R}^2$.

(b) Let S be the set of all $(x, y) \in \mathbb{R}^2$ at which $f(x, y) = 0$. Find those points of S that have no neighborhoods in which the equation $f(x, y) = 0$ can be solved for y in terms of x (or for x in terms of y). Describe S as precisely as you can.

8.6. Compare

$$\int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3} dy \right] dx \quad \text{and} \quad \int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3} dx \right] dy.$$

Comment on your results.

8.7. Bonus problem. Consider the equation

$$xe^y + ye^x = 0. \tag{1}$$

(a) Use a mathematical software to graph the function $f(x, y) = xe^y + ye^x$.

(b) Observe that there is no way to write down an explicit solution $y = y(x)$ of (1) in a neighborhood of the point $(x_0, y_0) = (0, 0)$.

(c) Why, nevertheless, does there exist a C^∞ solution $y = y(x)$ of (1) in a neighborhood of the point $(x_0, y_0) = (0, 0)$.

(d) What is its derivative at $x_0 = 0$?

(e) What is its second derivative at $x_0 = 0$?

(f) What does this tell you about the graph of the solution?