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Spring Term 2008

Analysis II — Problem Set 9

Issued: 15.03.08 Due: 23.03.08

9.1. Lagrange Multipliers.

Let $f, h \in C^1(U)$, $U \subseteq \mathbb{R}^n$ open, be given with $\nabla f(a) = (Df)_a \neq 0$ and $h(a) > h(x)$ for all $x \in B_\epsilon(a) \cap \{x : f(x) = 0\}$, $\epsilon > 0$. Then $\nabla f(a) = \lambda \nabla g(a)$ for some $\lambda \in \mathbb{R}$. Such λ is called *Lagrange Multiplier*.

Hint: W.l.o.g., assume that $\frac{\partial f}{\partial x_n} \neq 0$ and consider $U \subseteq \mathbb{R}^{n-1} \times \mathbb{R}$. The implicit function theorem provides a function g. Differentiation of the equation $f(x_1, \ldots, x_{n-1}, g(x_1, \ldots, x_{n-1})) = 0$ and the function $H(x_1, \ldots, x_{n-1}) = h(x_1, \ldots, x_{n-1}, g(x_1, \ldots, x_{n-1}))$ at $(a_1, \ldots, a_{n-1})$ provides two equations which combine to proof the result.

9.2. Find maxima and minima of the function $f(x, y) = 4x^2 - 3xy$ on the disk $D = \{(x, y) : \|(x, y)\| \leq 1\}$.
Hint: Consider first extrema on D° and then on ∂D.

9.3. Admissible sets.
A set $E \in \mathbb{R}^n$ is called admissible, if it is bounded (i.e. contained in some rectangle R) and if it's boundary ∂E is a zero set. The volume of an admissible set is defined to be

$$\mathrm{vol}\,(E) := \int_E 1 \cdot dx := \int_{R \supset E} \chi_E(x) dx,$$

where χ_E is a characteristic function.
Let $E \in \mathbb{R}^n$ be some admissible set, contained in some rectangle R. Show that

$$\mathrm{vol}\,(E) = \sup\{\sum_{\substack{I \subset E \\ I \in G}} |I| \ : \ G \text{ is a grid on } R,\ I \text{ are subrectangles of the grid } G\}.$$

Also prove: Given $\epsilon > 0$, there exist δ such that if $\mathrm{mesh}\, G < \delta$, then

$$\left| \mathrm{vol}\,(E) - \sum_{I \subset E} |I| \right| < \epsilon,$$

the sum being taken over all subrectangles I of G contained in E. Finally, prove that

$$\mathrm{vol}\,(E) = \inf_P \sum_{I \cap E \neq \varnothing} |I|,$$

the sum now being taken over all subrectangles I of the grid G having a non-empty intersection with E.
Hint: Use Darboux sums and Darboux integrals with respect to the function χ_E.

9.4. Integral over a set.
Riemann integral of a function over an admissible set E is defined to be

$$\int_E f(x)dx := \int_{R \supset E} f\chi_E(x)dx,$$

where R is an arbitrary rectangle, containing E. Assuming that $m \leq f(x) \leq M$ for any $x \in E$, show that

$$m \cdot \mathrm{vol}\,(E) \leq \int_E f(x)dx \leq M \cdot \mathrm{vol}\,(E).$$

Given additionally that f is continuous on E, prove that

$$\lim_{n \longrightarrow \infty} \left(\int_E f^n(x)dx \right)^{1/n} = \sup_{x \in E} f(x).$$

9.5. In the derivation of Fubini's theorem it is observed that for all $y \in [c, d] \subset Y$, where Y is a zero set, the lower and upper integrals with respect to x agree, $\underline{F}(y) = \overline{F}(y)$. One might think that the values of $\underline{F}$ and $\overline{F}$ on Y have no effect on their integrals. Not so. Consider the function defined on the unit square $[0, 1] \times [0, 1]$,

$$f(x, y) = \begin{cases} 1 & \text{if } y \text{ is irrational,} \\ 1 & \text{if } y \text{ is rational and } x \text{ is irrational,} \\ 1 - \frac{1}{q} & \text{if } y \text{ is rational and } x = \frac{p}{q} \text{ is rational and written in lowest terms.} \end{cases}$$

(a) Show that f is Riemann integrable, and its integral is 1.

(b) Observe that if $Y = \mathbb{Q} \cap [0, 1]$, then for each y which is not in Y,

$$\int_0^1 f(x, y)dx$$

exists and equals 1.

(c) Observe that if for each $y \in Y$ we choose in a completely arbitrary manner some $h(y) \in [\underline{F}(y), \overline{F}(y)]$ and set

$$H(y) = \begin{cases} \underline{F}(y) = \overline{F}(y) & \text{if } y \text{ not in } Y, \\ h(y) & \text{if } y \in Y. \end{cases}$$

then the integral of H exists and equals 1, but if we take

$$G(y) = \begin{cases} \underline{F}(y) = \overline{F}(y) & \text{if } y \text{ not in } Y, \\ 0 & \text{if } y \in Y. \end{cases}$$

the integral does not exist.

9.6. Bonus problem. Let $A \subset \mathbb{R}^2$ be a Jordan measurable set in the xy-plane contained in the half-plane $\{(x, y) : y \geq 0\}$ and let V be a subset of $\mathbb{R}^3$ obtained by revol ving the set A about the $x-$axis. Prove that the vol ume of V is equal to $\int_V 2\pi y dx dy$.