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Functional Analysis — Final Exam
26.5.10

Points achieved beyond 100 are Bonus Points.

F1. Let P be a bounded map on the normed space X satisfying $P^2 = P$.

(a) Show that either $\|P\| = 0$ or $\|P\| \geq 1$.

(b) Show that $\ker P$, $\operatorname{range} P$ are closed.

(c) Show that $X \cong \ker P \oplus \operatorname{range} P$, that is, any $x \in X$ can be written uniquely as $y + z$ with $y \in \ker P$ and $z \in \operatorname{range} P$, and X is topological isomorph to $\ker P + \operatorname{range} P$, each equipped with the relative topology.
(For the topological part, you may want to show that $x_n = y_n + z_n \longrightarrow x = y + z$ in X if and only if $y_n \longrightarrow y$ in $\ker P$ and $z_n \longrightarrow z$ in $\operatorname{range} P$.)

(d) Show that if X is a separable Hilbert space and U a closed linear subspace of X, then exists a bounded map Q on X with $Q^2 = Q$ and $\operatorname{range} Q = U$. (Note, this assertion does not necessarily hold in Banach spaces.) (25)

F2. Let $\|\cdot\|$ and $|||\cdot|||$ be two norms on the vector space X, both making X a Banach space. Show that if for some $M > 0$ we have $\|x\| \leq M|||x|||$ for all $x \in X$, then both norms are equivalent. (20)

F3. Let X be a Banach space, $S, T \in \mathcal{B}(X)$, then for $\lambda \in \rho(S) \cap \rho(T)$ we have

$$R_\lambda(S) - R_\lambda(T) = R_\lambda(S)(S - T)R_\lambda(T).$$

(20)

F4. Let H_1 and H_2 be separable Hilbert spaces and let $T : H_1 \longrightarrow H_2$ be a compact operator.

(a) Show that T^*T is a compact, positive, and self adjoint on H_1.

(b) For any compact, positive, self adjoint operator A on a Hilbert space exists a unique positive, self adjoint operator B with $B^2 = A$. Denote B by $A^{\frac{1}{2}}$.

(c) Let $|T| = (T^*T)^{\frac{1}{2}}$ and define $U \in \mathcal{B}(\operatorname{range}|T|, \operatorname{range} T)$ by setting $U(|T|x) = Tx$ for $y = |T|x \in \operatorname{range}|T|$. Show that U is isometric and extends to $U' \in \mathcal{B}(\overline{\operatorname{range}|T|}, \overline{\operatorname{range} T})$, and, by setting $U'x = 0$ for $x \in (\overline{\operatorname{range}|T|})^\perp$, we obtain (the polar decomposition) $T = U|T|$.
(Hint: First show $\|\,|T|x\| = \|Tx\|$ for all x.)

(d) Show that there exists an ONB $\{e_1, e_2, e_3, \ldots\}$ of H_1 and an ONB $\{f_1, f_2, f_3, \ldots\}$ of H_2, as well as a sequence (of so-called singular values) $s_1 \geq s_2 \geq s_3 \geq \ldots \geq 0$ with $s_k \to 0$ as $k \to \infty$ and

$$Tx = \sum_{k=1}^{\infty} s_k \langle x, e_k \rangle f_k, \quad x \in H_1.$$

(30)

F5. Let X be a reflexive Banach space.

(a) Show that $\operatorname{ball} X = \{x : \ \|x\| \leq 1\}$ is weakly compact, that is, compact with respect to the topology $\sigma(X, X^*)$.

(b) Show that every weakly Cauchy sequence, that is, a sequence $\{x_n\}$ with $\{\langle x_n, x^* \rangle\}$ is Cauchy in $\mathbb{F}$ for all $x^* \in X^*$, converges weakly to some $x \in X$. (30)