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Spring Term 2010

Functional Analysis —Midterm 1

22.3.10

MT1. Let H be a separable Hilbert space. Show that for $A : H \longrightarrow H$ hermitian, we have $I + iA$ is invertible.

MT2. Show that the space $l^1(\mathbb{N})$ is of first category in $l^2(\mathbb{N})$.

MT3. Let $w \in C(\mathbb{R})$ be real valued and let $\langle f, g\rangle_w = \int f(x)\overline{g(x)}w(x)\,dx$. Define a Hilbert space H_w using this sesquilinear form. Under which conditions is this in fact a Hilbert space? Under which conditions does it equal $L^2(\mathbb{R})$, that is, under which conditions does it define an equivalent norm?

MT4. Let X be a Banach space and $U \subseteq X$ closed. Let $S \in \mathbb{B}(U, l^\infty(\mathbb{N}))$. Show that there exists $T \in \mathbb{B}(X, l^\infty(\mathbb{N}))$ with T restricted to U is S and $\|T\| = \|S\|$. (Tip, apply Hahn Banach to $l_n : u \mapsto (Su)_n$.)

MT5. For X, Y normed spaces, we say that a bounded linear map $T : X \longrightarrow Y$ is Fredholm if

(a) $Kernel(T)$ is finite dimensional.

(b) $Image(T)$ is closed and finite codimensional.

Show that the right shift and the left shift operators on $l^2(\mathbb{N})$ are Fredholm. Moreover, show that for any compact operator $u : X \longrightarrow Y$ we have $I - u$ is Fredholm.