

## SINGULAR EQUIVARIANT ASYMPTOTICS AND THE MOMENT MAP I

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## 1. INTRODUCTION

This is the first of a series of papers dealing with the asymptotic behavior of certain integrals occurring in the description of the spectrum of an invariant elliptic operator on a compact Riemannian manifold  $M$  carrying the action of a compact, connected Lie group of isometries  $G$ , and in the study of its equivariant cohomology via the moment map  $\mathbb{J} : T^*M \rightarrow \mathfrak{g}^*$ , where  $T^*M$  and  $\mathfrak{g}$  denote the cotangent bundle of  $M$  and the Lie algebra of  $G$ , respectively. In the latter context [6, 1, 14, 2], the mentioned integrals are of the form

$$I(\mu) = \int_{T^*M \times \mathfrak{g}} e^{i\mathbb{J}(m)(X)/\mu} a(m, X) dm dX, \quad \mu \rightarrow 0^+,$$

where  $dm$  is a density on  $T^*M$ ,  $dX$  is the Lebesgue measure in  $\mathfrak{g}$ , and  $a \in C_c^\infty(T^*M \times \mathfrak{g})$  is an amplitude. While asymptotics for  $I(\mu)$  have been obtained for free group actions, one meets with serious difficulties when singular orbits are present. The reason is that, when trying to examine these integrals via the generalized stationary phase theorem in the case of general effective actions, the critical set of the phase function  $\mathbb{J}(m)(X)$  is no longer a smooth manifold, so that, a priori, the principle of the stationary phase can not be applied in this case. Nevertheless, in what follows, we shall show how to circumvent this obstacle by partially resolving the singularities of the critical set of  $\mathbb{J}(m)(X)$ , and in this way obtain asymptotics for  $I(\mu)$  with remainder estimates in the case of singular group actions. We shall restrict ourselves first to orthogonal group actions in  $\mathbb{R}^n$ , while the global theory, together with the applications, shall be treated in a second paper.

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## 2. COMPACT GROUP ACTIONS AND THE MOMENT MAP

From now on let  $G$  be a compact, connected Lie group with Lie algebra  $\mathfrak{g}$  acting orthogonally on Euclidean space  $\mathbb{R}^n$ . Note that any finite-dimensional, separable metric  $G'$ -space with only finitely many orbit types, where  $G'$  is a compact Lie group, may be embedded equivariantly in an orthogonal action of  $G'$  on some Euclidean space, see Bredon [3], Section II.10. The considered type of group actions is therefore already quite general. Consider now the cotangent space  $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$ , endowed with the canonical coordinates  $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ . It constitutes a symplectic manifold whose symplectic form is given by

$$\omega = d\theta = \sum_{i=1}^n d\xi_i \wedge dx_i,$$

where  $\theta = \sum \xi_i dx_i$  is the Liouville form. The group  $G$  acts on  $T^*\mathbb{R}^n$  by  $g(x, \xi) = (gx, g\xi)$  in a Hamiltonian way, and if we denote by  $\tilde{X}$  the fundamental vector field generated by an element  $X$  of  $\mathfrak{g}$ , the corresponding moment map is given by

$$\mathbb{J} : T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathfrak{g}^*, \quad \mathbb{J}(x, \xi)(X) = \theta(\tilde{X})(x, \xi) = \langle Xx, \xi \rangle,$$

where  $\langle \cdot, \cdot \rangle$  is the Euclidean inner product in  $\mathbb{R}^n$ . We are interested in the asymptotic behavior of integrals of the form

$$(1) \quad I(\mu) = \int_{T^*\mathbb{R}^n} \int_{\mathfrak{g}} e^{i\psi(x, \xi, X)/\mu} a(x, \xi, X) dX d\xi dx, \quad \mu \rightarrow 0^+,$$

where  $dx d\xi$ , and  $dX$  are Lebesgue measures in  $T^*\mathbb{R}^n$ , and  $\mathfrak{g}$ , respectively,  $a \in C_c^\infty(T^*\mathbb{R}^n \times \mathfrak{g})$ , and

$$\psi(x, \xi, X) = \mathbb{J}(x, \xi)(X) = \langle Xx, \xi \rangle.$$

We would like to study the integrals  $I(\mu)$  by means of the generalized stationary phase theorem, and for this we have to consider the critical set of the phase function  $\psi(x, \xi, X)$  given by

$$\text{Crit}(\psi) = \{(x, \xi, X) \in T^*\mathbb{R}^n \times \mathfrak{g} : \psi_*(x, \xi, X) = 0\} = \{(x, \xi, X) \in \Omega \times \mathfrak{g} : X \in \mathfrak{g}_{(x, \xi)}\},$$

where

$$\Omega = \mathbb{J}^{-1}(0) = \{(x, \xi) \in T^*\mathbb{R}^n : \langle Ax, \xi \rangle = 0 \text{ for all } A \in \mathfrak{g}\}$$

represents the zero level of the moment map, and

$$\mathfrak{g}_{(x, \xi)} = \{X \in \mathfrak{g} : Xx = 0, \quad X\xi = 0\}$$

denotes the Lie algebra of the isotropy group  $G_{(x, \xi)}$  stabilizing the point  $(x, \xi)$ . Now, the major difficulty in applying the generalized stationary phase theorem in our setting stems from the fact that, due to the singular orbit structure of the underlying group action, the zero level  $\Omega$  of the moment map, and, consequently, the considered critical set  $\text{Crit}(\psi)$ , are in general singular varieties. In fact, if the  $G$ -action on  $T^*\mathbb{R}^n$  is not free, the considered moment map is no longer a submersion, so that  $\Omega$  and the symplectic quotient  $\Omega/G$  are not smooth anymore. Nevertheless, it can be shown that these spaces have Whitney stratifications into smooth submanifolds, see [13], and Ortega-Ratiu [12], Theorems 8.3.1 and 8.3.2, which correspond to the stratifications of  $T^*\mathbb{R}^n$ , and  $\mathbb{R}^n$  by orbit types, see Duistermaat-Kolk [7]. In particular,  $\Omega$  has a principal stratum given by

$$\text{Reg } \Omega = \{(x, \xi) \in \Omega : G_{(x, \xi)} \text{ is of principal type}\},$$

which is an open and dense subset of  $\Omega$ , see Cassanas-Ramacher [4], Proposition 2. In addition,  $\text{Reg } \Omega$  is a smooth submanifold in  $\mathbb{R}^{2n}$  of codimension equal to the dimension  $\kappa$  of a principal orbit. It is then clear that the smooth part of  $\text{Crit}(\psi)$  is given by

$$\text{Reg } \text{Crit}(\psi) = \{(x, \xi, X) \in \text{Reg } \Omega \times \mathfrak{g} : X \in \mathfrak{g}_{(x, \xi)}\},$$

and constitutes a submanifold of codimension  $2\kappa$ . To obtain an asymptotic description of  $I(\mu)$ , we shall partially resolve the singularities of  $\text{Crit}(\psi)$ , for which we will need a suitable  $G$ -invariant covering of  $\mathbb{R}^n$ . More generally, and following Kawakubo [9], Theorem 4.20, we shall construct such a covering for an arbitrary compact Riemannian  $K$ -manifold  $M$ , where  $K$  is a compact, connected Lie group of isometries. Thus, let  $(H_1), \dots, (H_L)$  denote the isotropy types of  $M$ , and arrange them in such a way that

$$H_j \text{ is conjugated to a subgroup of } H_i \quad \Rightarrow \quad i \leq j.$$

Let  $H \subset K$  be a closed subgroup, and  $M(H)$  the union of all orbits of type  $K/H$ . Then  $M$  has a stratification into orbit types according to

$$M = M(H_1) \cup \dots \cup M(H_L).$$

By the principal orbit theorem, the set  $M(H_L)$  is open and dense in  $M$ , while  $M(H_1)$  is a closed,  $K$ -invariant submanifold. Denote by  $\nu_1$  the normal  $K$ -vector bundle of  $M(H_1)$ , and by  $f_1 : \nu_1 \rightarrow M$  a  $K$ -invariant tubular neighbourhood of  $M(H_1)$  in  $M$ . Take a  $K$ -invariant metric on  $\nu_1$ , and put

$$D_t(\nu_1) = \{v \in \nu_1 : \|v\| \leq t\}, \quad t > 0.$$

We then define the compact,  $K$ -invariant submanifold with boundary

$$M_2 = M - f_1(\overset{\circ}{D}_{1/2}(\nu_1)),$$

on which the isotropy type  $(H_1)$  no longer occurs, and endow it with a  $K$ -invariant Riemannian metric with product form in a  $K$ -invariant collar neighborhood of  $\partial M_2$  in  $M_2$ . Consider now the union  $M_2(H_2)$  of orbits in  $M_2$  of type  $K/H_2$ , a compact  $K$ -invariant submanifold of  $M_2$  with boundary, and let  $f_2 : \nu_2 \rightarrow M_2$  be a  $K$ -invariant tubular neighbourhood of  $M_2(H_2)$  in  $M_2$ , which exists due to the particular form of the metric on  $M_2$ . Taking a  $K$ -invariant metric on  $\nu_2$ , we define

$$M_3 = M_2 - f_2(\overset{\circ}{D}_{1/2}(\nu_2)),$$

which constitutes a compact  $K$ -invariant submanifold with corners and isotropy types  $(H_3), \dots, (H_L)$ . Continuing this way, one finally obtains for  $M$  the decomposition

$$M = f_1(D_{1/2}(\nu_1)) \cup \dots \cup f_L(D_{1/2}(\nu_L)),$$

where we identified  $f_L(D_{1/2}(\nu_L))$  with  $M_L$ , which leads to the covering

$$M = f_1(\overset{\circ}{D}_1(\nu_1)) \cup \dots \cup f_L(\overset{\circ}{D}_1(\nu_L)), \quad f_L(\overset{\circ}{D}_1(\nu_L)) = \overset{\circ}{M}_L.$$

In exactly the same way, one shows the existence of a covering

$$\mathbb{R}^n = f_1(\overset{\circ}{D}_1(\nu_1)) \cup \dots \cup f_L(\overset{\circ}{D}_1(\nu_L))$$

of  $\mathbb{R}^n$  by  $G$ -invariant tubular neighbourhoods, where  $f_L(\overset{\circ}{D}_1(\nu_L)) \equiv \overset{\circ}{\mathbb{R}}^n_L$ , the notation being as before.

### 3. THE DESINGULARIZATION PROCESS

Let us now start resolving the singularities of the critical set  $\text{Crit}(\psi)$ . For this, we will have to set up an iterative desingularization process along the strata of the underlying  $G$ -action, where each step in our iteration will consist of a decomposition, a monoidal transformation, and a reduction. For simplicity, we shall assume that at each iteration step the set of maximally singular orbits is connected. Otherwise each of the connected components, which might even have different dimensions, has to be treated separately.

**First decomposition.** As before, let  $f_k : \nu_k \rightarrow M_k$  be an invariant tubular neighborhood of  $M_k(H_k)$  in

$$M_k = \mathbb{R}^n - \bigcup_{i=1}^{k-1} f_i(\overset{\circ}{D}_{1/2}(\nu_i)),$$

a manifold with corners on which  $G$  acts with the isotropy types  $(H_k), (H_{k+1}), \dots, (H_L)$ , and put  $W_k = f_k(\overset{\circ}{D}_1(\nu_k))$ . Introduce a partition of unity  $\{\chi_k\}_{k=1, \dots, L}$  subordinated to the covering  $\{W_k\}$ , and define

$$(2) \quad I_k(\mu) = \int_{T^*\mathbb{R}^n} \int_{\mathfrak{g}} e^{i\psi(x, \xi, X)/\mu} a(x, \xi, X) \chi_k(x) dX d\xi dx,$$

so that  $I(\mu) = I_1(\mu) + \dots + I_L(\mu)$ . Now, if  $(x, \xi) \in \Omega$ , and either  $x$  or  $\xi$  belong to  $\mathbb{R}^n(H_L)$ , by Cassanas-Ramacher [4], Proposition 2, it follows already that  $(x, \xi) \in \text{Reg } \Omega$ . The critical set of  $\psi$  is therefore a smooth manifold in a neighborhood of  $\text{supp } \chi_L a$ , since  $f_L(\overset{\circ}{D}_1(\nu_L)) \subset \mathbb{R}^n(H_L)$ . Furthermore, it is clear that the transversal Hessian of  $\psi$  is non-degenerate on  $\text{Reg Crit}(\psi)$ . For this reason, the stationary phase theorem can directly be applied to compute the integral  $I_L(\mu)$ . Let us therefore turn to the case that  $k \in \{1, \dots, L-1\}$ . The sets

$$\begin{aligned} \Omega_k &= \{(x, \xi) \in W_k \times \mathbb{R}^n : \langle Ax, \xi \rangle = 0 \text{ for all } A \in \mathfrak{g}\}, \\ \text{Crit}_k(\psi) &= \{(x, \xi, X) \in \Omega_k \times \mathfrak{g} : X \in \mathfrak{g}_{(x, \xi)}\} \end{aligned}$$

are then no longer smooth manifolds, and since  $\text{supp } \chi_k \subset W_k$ , the stationary phase theorem can not be applied directly in this situation. Instead, we shall resolve the singularities of  $\text{Crit}_k(\psi)$ , and after this apply the principle of the stationary phase in a suitable resolution space. For this, introduce for each  $p^{(k)} \in M_k(H_k)$  the decomposition

$$\mathfrak{g} = \mathfrak{g}_{p^{(k)}} \oplus \mathfrak{g}_{p^{(k)}}^\perp,$$

where  $\mathfrak{g}_{p^{(k)}}$  denotes the Lie algebra of stabilizer  $G_{p^{(k)}}$  of  $p^{(k)}$ , and  $\mathfrak{g}_{p^{(k)}}^\perp$  its orthogonal complement with respect to the scalar product  $\text{tr}({}^t AB)$  in  $\mathfrak{g}$ . Let further  $A_1(p^{(k)}), \dots, A_{d^{(k)}}(p^{(k)})$  be an orthonormal basis of  $\mathfrak{g}_{p^{(k)}}^\perp$ , and  $B_1(p^{(k)}), \dots, B_{e^{(k)}}(p^{(k)})$  an orthonormal basis of  $\mathfrak{g}_{p^{(k)}}$ . Consider the isotropy algebra bundle over  $M_k(H_k)$

$$\text{iso } M_k(H_k) \rightarrow M_k(H_k),$$

as well as the canonical projection

$$\pi_k : W_k \rightarrow M_k(H_k), \quad f_k(p^{(k)}, v^{(k)}) \mapsto p^{(k)}, \quad p^{(k)} \in M_k(H_k), v^{(k)} \in (\nu_k)_{p^{(k)}},$$

where  $f_k(p^{(k)}, v^{(k)}) = (\exp_{p^{(k)}} \circ \gamma^{(k)})(v^{(k)})$ , and  $\gamma^{(k)}$  is some scaling function, see [3], page 306-307. We then consider the induced bundle

$$\pi_k^* \text{iso } M_k(H_k) = \left\{ (f_k(p^{(k)}, v^{(k)}), X) \in W_k \times \mathfrak{g} : X \in \mathfrak{g}_{p^{(k)}} \right\},$$

and denote by

$$\Pi_k : W_k \times \mathfrak{g} \rightarrow \pi_k^* \text{iso } M_k(H_k)$$

the canonical projection which is obtained by considering geodesic normal coordinates around  $\pi_k^* \text{iso } M_k(H_k)$ , and identifying  $W_k \times \mathfrak{g}$  with a neighborhood of the zero section in the normal bundle  $N \pi_k^* \text{iso } M_k(H_k)$ . Note also that the fiber of the normal bundle to  $\pi_k^* \text{iso } M_k(H_k)$  at a point

$(f_k(p^{(k)}, v^{(k)}), X)$  can be identified with  $\mathfrak{g}_{p^{(k)}}^\perp$ . Integrating along the fibers of the normal bundle to  $\pi_k^* \mathfrak{iso} M_k(H_k)$  we therefore obtain for  $I_k(\mu)$  the expression

$$\begin{aligned} I_k(\mu) &= \int_{\pi_k^* \mathfrak{iso} M_k(H_k)} \left[ \int_{\Pi_k^{-1}(p^{(k)}, v^{(k)}, B^{(k)}) \times \mathbb{R}^n} e^{i\psi/\mu} \chi_k a \Phi_k d\xi dA^{(k)} \right] dB^{(k)} dv^{(k)} dp^{(k)} \\ &= \int_{M_k(H_k)} \left[ \int_{\mathfrak{g} \times \pi_k^{-1}(p^{(k)}) \times \mathbb{R}^n} e^{i\psi/\mu} \chi_k a \Phi_k d\xi dA^{(k)} dB^{(k)} dv^{(k)} \right] dp^{(k)}, \end{aligned}$$

where

$$\gamma^{(k)}(\overset{\circ}{D}_1(\nu_k)_{p^{(k)}}) \times \mathfrak{g}_{p^{(k)}}^\perp \times \mathfrak{g}_{p^{(k)}} \ni (v^{(k)}, A^{(k)}, B^{(k)}) \mapsto (\exp_{p^{(k)}} v^{(k)}, A^{(k)} + B^{(k)}) = (x, X)$$

are coordinates on  $\mathfrak{g} \times \pi_k^{-1}(p^{(k)})$ , while  $dp^{(k)}$ , and  $dA^{(k)}, dB^{(k)}, dv^{(k)}$  are suitable measures in  $M_k(H_k)$ , and  $\mathfrak{g}_{p^{(k)}}^\perp, \mathfrak{g}_{p^{(k)}}, \overset{\circ}{D}_1(\nu_k)_{p^{(k)}}$ , respectively, such that  $dX dx \equiv \Phi_k dA^{(k)} dB^{(k)} dv^{(k)} dp^{(k)}$ .

**First monoidal transformation.** Let now  $k \in \{1, \dots, L-1\}$  be fixed. For the further analysis of the integral  $I_k(\mu)$ , we shall successively resolve the singularities of  $\text{Crit}_k(\psi)$ , until we are in position to apply the principle of the stationary phase in a suitable resolution space. To begin with, we perform a monoidal transformation

$$\zeta_k : B_{Z_k}(W_k \times \mathfrak{g}) \longrightarrow W_k \times \mathfrak{g}$$

in  $W_k \times \mathfrak{g}$  with center  $Z_k = \mathfrak{iso} M_k(H_k)$ . For this, let us write  $A^{(k)}(p^{(k)}, \alpha^{(k)}) = \sum \alpha_i^{(k)} A_i^{(k)}(p^{(k)})$ ,  $B^{(k)}(p^{(k)}, \beta^{(k)}) = \sum \beta_i^{(k)} B_i^{(k)}(p^{(k)})$ , and

$$v^{(k)}(p^{(k)}, \theta^{(k)}) = \sum_{i=1}^{c^{(k)}} \theta_i^{(k)} v_i^{(k)}(p^{(k)}),$$

where  $\{v_1^{(k)}(p^{(k)}), \dots, v_{c^{(k)}}^{(k)}(p^{(k)})\}$  denotes an orthonormal frame in  $\nu_k$ . With respect to these coordinates we have  $Z_k = \{\alpha^{(k)} = 0, \theta^{(k)} = 0\}$ , so that

$$\begin{aligned} B_{Z_k}(W_k \times \mathfrak{g}) &= \left\{ (x, X, [t]) \in W_k \times \mathfrak{g} \times \mathbb{R}\mathbb{P}^{c^{(k)}+d^{(k)}-1} : \theta_i^{(k)} t_j = \theta_j^{(k)} t_i, \alpha_i^{(k)} t_{c^{(k)}+j} = \alpha_j^{(k)} t_{c^{(k)}+i} \right\}, \\ \zeta_k : (x, X, [t]) &\longmapsto (x, X). \end{aligned}$$

If we now cover  $B_{Z_k}(W_k \times \mathfrak{g})$  with the charts  $\{(\varphi_\varrho, U_\varrho)\}$ ,  $U_\varrho = B_{Z_k}(W_k \times \mathfrak{g}) \cap (W_k \times \mathfrak{g} \times V_\varrho)$ , where  $V_\varrho = \left\{ [t] \in \mathbb{R}\mathbb{P}^{c^{(k)}+d^{(k)}-1} : t_\varrho \neq 0 \right\}$ , we obtain for  $\zeta_k$  in each of the  $\theta^{(k)}$ -charts  $\{U_\varrho\}_{1 \leq \varrho \leq c^{(k)}}$  the expressions

$${}^\varrho \zeta_k = \zeta_k \circ \varphi_\varrho : (p^{(k)}, \tau_k, {}^\varrho \tilde{v}^{(k)}, A^{(k)}, B^{(k)}) \mapsto (\exp_{p^{(k)}} \tau_k {}^\varrho \tilde{v}^{(k)}, \tau_k A^{(k)} + B^{(k)}) \equiv (x, X),$$

where

$${}^\varrho \tilde{v}^{(k)}(p^{(k)}, \theta^{(k)}) = \left( v_\varrho^{(k)}(p^{(k)}) + \sum_{i \neq \varrho} \theta_i^{(k)} v_i^{(k)}(p^{(k)}) \right) / \sqrt{1 + \sum_{i \neq \varrho} (\theta_i^{(k)})^2} \in ({}^\varrho S_k^+)_{p^{(k)}},$$

and

$${}^\varrho S_k^+ = \left\{ v \in \nu_k : v = \sum s_i v_i, s_\varrho > 0, \|v\| = 1 \right\}.$$

Note that for each  $1 \leq \varrho \leq c^{(k)}$ ,

$$W_k \simeq f_k({}^\varrho S_k^+ \times (-1, 1))$$

up to a set of measure zero, and since  $f_k(p^{(k)}, v^{(k)}) = (\exp_{p^{(k)}} \circ \gamma^{(k)})(v^{(k)})$ , we have  $\tau_k \in (-T, T)$  for some  $1 > T > 0$ . As a consequence, we obtain for the phase function the factorization

$$\begin{aligned} \psi(x, \xi, X) &= \left\langle (\tau_k A^{(k)} + B^{(k)}) \exp_{p^{(k)}} \tau_k \varrho \tilde{v}^{(k)}, \xi \right\rangle \\ &= \tau_k \left[ \left\langle A^{(k)} p^{(k)} + B^{(k)} \varrho \tilde{v}^{(k)}, \xi \right\rangle + \tau_k \left\langle A^{(k)} \varrho \tilde{v}^{(k)}, \xi \right\rangle \right], \end{aligned}$$

where we took into account that  $\exp_{p^{(k)}} v^{(k)} = p^{(k)} + v^{(k)}$ . Similar considerations hold for  $\zeta_k$  in the  $\alpha^{(k)}$ -charts  $\{U_\varrho\}_{c^{(k)}+1 \leq \varrho \leq c^{(k)}+d^{(k)}}$ , so that we get

$$\psi \circ (\text{id}_\xi \otimes \zeta_k) = {}^{(k)}\tilde{\psi}^{tot} = \tau_k \cdot {}^{(k)}\tilde{\psi}^{wk},$$

${}^{(k)}\tilde{\psi}^{tot}$  and  ${}^{(k)}\tilde{\psi}^{wk}$  being the *total* and *weak transform* of the phase function  $\psi$ , respectively.<sup>1</sup> Introducing a partition  $\{u_\varrho\}$  of unity subordinated to the covering  $\{U_\varrho\}$  now yields

$$I_k(\mu) = \sum_{\varrho=1}^{c^{(k)}} \varrho I_k(\mu) + \sum_{\varrho=c^{(k)}+1}^{d^{(k)}} \varrho \tilde{I}_k(\mu),$$

where the integrals  $\varrho I_k(\mu)$  and  $\varrho \tilde{I}_k(\mu)$  are given by the expressions

$$\int_{M_k(H_k)} \left[ \int_{\varrho \zeta_k^{-1}(\mathfrak{g} \times \pi_k^{-1}(p^{(k)})) \times \mathbb{R}^n} (u_\varrho \circ \varphi_\varrho) (\varrho \zeta_k)^* (e^{i\psi/\mu} \chi_k a \Phi_k d\xi dA^{(k)} dB^{(k)} dv^{(k)}) d\xi \right] dp^{(k)}.$$

As we shall see in Section 8, the weak transform  ${}^{(k)}\tilde{\psi}^{wk}$  has no critical points in the  $\alpha^{(k)}$ -charts, which will imply that the integrals  $\varrho \tilde{I}_k(\mu)$  contribute to  $I(\mu)$  only with higher order terms. In what follows, we shall therefore restrict ourselves to the situation where  $\chi_k a \circ (\text{id}_\xi \otimes \zeta_k)$  has compact support in one of the  $\theta^{(k)}$ -charts. Thus we can assume  $I_k(\mu)$  to be given by

$$\begin{aligned} & \int_{M_k(H_k)} \left[ \int_{\zeta_k^{-1}(\mathfrak{g} \times \pi_k^{-1}(p^{(k)})) \times \mathbb{R}^n} e^{i\frac{\tau_k}{\mu} {}^{(k)}\tilde{\psi}^{wk}} (\chi_k a \circ (\text{id}_\xi \otimes \zeta_k)) \tilde{\Phi}_k d\xi dA^{(k)} dB^{(k)} d\tilde{v}^{(k)} d\tau_k \right] dp^{(k)} \\ &= \int_{M_k(H_k) \times (-T, T)} \left[ \int_{(S_k^+)_p^{(k)} \times \mathfrak{g}_{p^{(k)}} \times \mathfrak{g}_{p^{(k)}}^\perp \times \mathbb{R}^n} e^{i\frac{\tau_k}{\mu} {}^{(k)}\tilde{\psi}^{wk}} (\chi_k a \circ (\text{id}_\xi \otimes \zeta_k)) \tilde{\Phi}_k \right. \\ & \quad \left. d\xi dA^{(k)} dB^{(k)} d\tilde{v}^{(k)} \right] d\tau_k dp^{(k)}, \end{aligned}$$

where we skipped the index  $\varrho$ , and took into account that

$$\zeta_k^{-1}(\mathfrak{g} \times \pi_k^{-1}(p^{(k)})) = \{p^{(k)}\} \times (-T, T) \times (S_k^+)_{p^{(k)}} \times \mathfrak{g}_{p^{(k)}} \times \mathfrak{g}_{p^{(k)}}^\perp.$$

Here  $d\tilde{v}^{(k)}$  is a suitable measure on  $(S_k^+)_{p^{(k)}}$  such that  $dX dx \equiv \tilde{\Phi}_k dA^{(k)} dB^{(k)} d\tilde{v}^{(k)} d\tau_k dp^{(k)}$ . Furthermore, a computation shows that

$$\tilde{\Phi}_k = |\tau_k|^{c^{(k)}+d^{(k)}-1} \Phi_k \circ \zeta_k.$$

**First reduction.** Let us now assume that there exists a  $x \in W_k$  with orbit type  $G/H_j$ , and let  $p^{(k)} \in M_k(H_k)$ ,  $v^{(k)} \in (\nu_k)_{p^{(k)}}$  be such that  $x = f_k(p^{(k)}, v^{(k)})$ . Since  $x$  lies in a slice at  $p$  around the  $G$ -orbit of  $p^{(k)}$ , we have  $G_x \subset G_{p^{(k)}}$  by Bredon [3], page 86. Hence,  $H_j \simeq G_x$  must be conjugated to a subgroup of  $H_k \simeq G_{p^{(k)}}$ . Now,  $G$  acts on  $M_k$  with the isotropy types  $(H_k), (H_{k+1}), \dots, (H_L)$ . The isotropy types occurring in  $W_k$  are therefore those for which the corresponding isotropy groups  $H_k, H_{k+1}, \dots, H_L$  are conjugated to a subgroup of  $H_k$ , and we shall denote them by

$$(H_k) = (H_{i_1}), (H_{i_2}), \dots, (H_L).$$

<sup>1</sup>For an explanation of this notation, see section 6.

Consequently,  $G$  acts on  $S_k$  with the isotropy types  $(H_{i_2}), \dots, (H_L)$ . Now, for every  $p^{(k)} \in M_k(H_k)$ ,  $(\nu_k)_{p^{(k)}}$  is an orthogonal  $G_{p^{(k)}}$ -space; furthermore, by the invariant tubular neighborhood theorem, one has the isomorphism

$$W_k/G \simeq (\nu_k)_{p^{(k)}}/G_{p^{(k)}}.$$

Therefore  $G_{p^{(k)}}$  acts on the manifold  $(S_k)_{p^{(k)}}$  with the isotropy types  $(H_{i_2}), \dots, (H_L)$  as well. As will turn out, if  $G$  acted on  $S_k$  only with type  $(H_L)$ , the critical set of  ${}^{(k)}\tilde{\psi}^{wk}$  would be clean in the sense of Bott, and we could proceed to apply the stationary phase theorem to compute  $I_k(\mu)$ . But in general this will not be the case, and we are forced to continue with the iteration.

**Second decomposition.** For every fixed  $p^{(k)} \in M_k(H_k)$  consider the covering of the compact  $G_{p^{(k)}}$ -manifold  $(S_k)_{p^{(k)}}$  given by

$$(S_k)_{p^{(k)}} = W_{ki_2} \cup \dots \cup W_{kL}, \quad W_{ki_j} = f_{ki_j}(\mathring{D}_1(\nu_{ki_j})), \quad W_{kL} = \text{Int}((S_k)_{p^{(k)},L}),$$

where  $f_{ki_j} : \nu_{ki_j} \rightarrow (S_k)_{p^{(k)},i_j}$  is an invariant tubular neighborhood of  $(S_k)_{p^{(k)},i_j}(H_{i_j})$  in

$$(S_k)_{p^{(k)},i_j} = (S_k)_{p^{(k)}} - \bigcup_{r=2}^{j-1} f_{ki_r}(\mathring{D}_{1/2}(\nu_{ki_r})), \quad j \geq 2,$$

and  $f_{ki_j}(p^{(i_j)}, v^{(i_j)}) = (\exp_{p^{(i_j)}} \circ \gamma^{(i_j)})(v^{(i_j)})$ ,  $p^{(i_j)} \in (S_k)_{p^{(k)},i_j}(H_{i_j})$ ,  $v^{(i_j)} \in (\nu_{ki_j})_{p^{(i_j)}}$ . Let further  $\{\chi_{ki_j}\}$  denote a partition of the unity subordinated to the covering  $\{W_{ki_j}\}$ , and define

$$I_{ki_j}(\mu) = \int_{M_k(H_k) \times (-T,T)} \left[ \int_{(S_k^+)_{p^{(k)}} \times \mathfrak{g}_{p^{(k)}} \times \mathfrak{g}_{p^{(k)}}^\perp \times \mathbb{R}^n} e^{i \frac{\tau_k}{\mu} {}^{(k)}\tilde{\psi}^{wk}} (\chi_k a \circ (\text{id}_\xi \otimes \zeta_k)) \chi_{ki_j} \tilde{\Phi}_k \right. \\ \left. d\xi dA^{(k)} dB^{(k)} d\tilde{v}^{(k)} \right] d\tau_k dp^{(k)},$$

so that  $I_k(\mu) = I_{ki_2}(\mu) + \dots + I_{kL}(\mu)$ . It is important to note that the partition functions  $\chi_{ki_j}$  depend smoothly on  $p^{(k)}$  as a consequence of the tubular neighborhood theorem, by which in particular  $S_k/G \simeq (S_k)_{p^{(k)}}/G_{p^{(k)}}$ , and the smooth dependence in  $p^{(k)}$  of the Riemannian metrics on the normal bundles  $\nu_{ki_j}$  and the manifolds with corners  $(S_k)_{p^{(k)},i_j}$ . Since  $G_{p^{(k)}}$  acts on  $W_{kL}$  only with type  $(H_L)$ , the iteration process for  $I_{kL}(\mu)$  ends here. For the remaining integrals  $I_{ki_j}(\mu)$  with  $k < i_j < L$ , let us denote by

$$\mathfrak{iso}(S_k)_{p^{(k)},i_j}(H_{i_j}) \rightarrow (S_k)_{p^{(k)},i_j}(H_{i_j})$$

the isotropy algebra bundle over  $(S_k)_{p^{(k)},i_j}(H_{i_j})$ , and by  $\pi_{ki_j} : W_{ki_j} \rightarrow (S_k)_{p^{(k)},i_j}(H_{i_j})$  the canonical projection. For  $p^{(i_j)} \in (S_k)_{p^{(k)},i_j}(H_{i_j})$ , consider the decomposition

$$\mathfrak{g} = \mathfrak{g}_{p^{(k)}} \oplus \mathfrak{g}_{p^{(k)}}^\perp = (\mathfrak{g}_{p^{(i_j)}} \oplus \mathfrak{g}_{p^{(i_j)}}^\perp) \oplus \mathfrak{g}_{p^{(k)}}^\perp.$$

Let further  $A_1^{(i_j)}, \dots, A_{d^{(i_j)}}^{(i_j)}$  be an orthonormal frame in  $\mathfrak{g}_{p^{(i_j)}}^\perp$ , as well as  $B_1^{(i_j)}, \dots, B_{e^{(i_j)}}^{(i_j)}$  be an orthonormal frame in  $\mathfrak{g}_{p^{(i_j)}}$ , and  $v_1^{(ki_j)}, \dots, v_{c^{(i_j)}}^{(ki_j)}$  an orthonormal frame in  $(\nu_{ki_j})_{p^{(i_j)}}$ . Integrating along the fibers in a neighborhood of  $\pi_{ki_j}^* \mathfrak{iso}(S_k)_{p^{(k)},i_j}(H_{i_j}) \subset W_{ki_j} \times \mathfrak{g}_{p^{(k)}}$  then yields for  $I_{ki_j}(\mu)$  the expression

$$I_{ki_j}(\mu) = \int_{M_k(H_k) \times (-T,T)} \left[ \int_{(S_k)_{p^{(k)},i_j}(H_{i_j})} \left[ \int_{\pi_{ki_j}^{-1}(p^{(i_j)}) \times \mathfrak{g}_{p^{(k)}} \times \mathfrak{g}_{p^{(k)}}^\perp \times \mathbb{R}^n} e^{i \frac{\tau_k}{\mu} {}^{(k)}\tilde{\psi}^{wk}} \right. \right. \\ \left. \left. \times (\chi_k a \circ (\text{id}_\xi \otimes \zeta_k)) \chi_{ki_j} \tilde{\Phi}_{ki_j} d\xi dA^{(k)} dA^{(i_j)} dB^{(i_j)} dv^{(i_j)} \right] dp^{(i_j)} \right] d\tau_k dp^{(k)},$$

where  $\Phi_{ki_j}$  is a Jacobian, and

$\gamma^{(i_j)}(\overset{\circ}{D}_1(\nu_{ki_j})_{p^{(i_j)}}) \times \mathfrak{g}_{p^{(i_j)}}^\perp \times \mathfrak{g}_{p^{(i_j)}} \ni (v^{(i_j)}, A^{(i_j)}, B^{(i_j)}) \mapsto (\exp_{p^{(i_j)}} v^{(i_j)}, A^{(i_j)} + B^{(i_j)}) = (\tilde{v}^{(k)}, B^{(k)})$  are coordinates on  $\pi_{ki_j}^{-1}(p^{(i_j)}) \times \mathfrak{g}_{p^{(k)}}$ , while  $dp^{(i_j)}$ , and  $dA^{(i_j)}, dB^{(i_j)}, dv^{(i_j)}$  are suitable measures in the spaces  $(S_k)_{p^{(k)}, i_j}(H_{i_j})$ , and  $\mathfrak{g}_{p^{(i_j)}}^\perp, \mathfrak{g}_{p^{(i_j)}}, \overset{\circ}{D}_1(\nu_{ki_j})_{p^{(i_j)}}$ , respectively, such that we have the equality  $\tilde{\Phi}_k dB^{(k)} d\tilde{v}^{(k)} \equiv \Phi_{ki_j} dA^{(i_j)} dB^{(i_j)} dv^{(i_j)} dp^{(i_j)}$ .

**Second monoidal transformation.** Let us fix an  $l$  such that  $k < l < L$ , and consider in the  $\theta^{(k)}$ -chart  $(-T, T) \times S_k^+ \times \mathfrak{g}$  a monoidal transformation

$$\zeta_{kl} : B_{Z_{kl}}((-T, T) \times S_k^+ \times \mathfrak{g}) \longrightarrow (-T, T) \times S_k^+ \times \mathfrak{g}$$

with center

$$Z_{kl} = (-T, T) \times \text{iso } S_{k,l}^+(H_l), \quad S_{k,l} = \bigcup_{p^{(k)} \in M_k(H_k)} (S_k)_{p^{(k)}, l}.$$

Writing  $A^{(l)}(p^{(l)}, \alpha^{(l)}) = \sum \alpha_i^{(l)} A_i^{(l)}(p^{(l)})$ ,  $B^{(l)}(p^{(l)}, \beta^{(l)}) = \sum \beta_i^{(l)} B_i^{(l)}(p^{(l)})$ , and

$$v^{(l)}(p^{(l)}, \theta^{(l)}) = \sum_{i=1}^{c^{(l)}} \theta_i^{(l)} v_i^{(kl)}(p^{(l)}),$$

one has  $Z_{kl} = \{\alpha^{(k)} = 0, \alpha^{(l)} = 0, \theta^{(l)} = 0\}$ . If we now cover  $B_{Z_{kl}}((-T, T) \times S_k^+ \times \mathfrak{g})$  with the standard charts, we shall see again in Section 8 that modulo higher order terms we can assume that  $((\chi_k a \circ (\text{id}_\xi \otimes \zeta_k))\chi_{kl}) \circ \zeta_{kl}$  has compact support in one of the  $\theta^{(l)}$ -charts. Therefore it suffices to examine  $\zeta_{kl}$  in one of these charts, in which it reads

$$\begin{aligned} \zeta_{kl} : (p^{(k)}, \tau_k, p^{(l)}, \tau_l, \tilde{v}^{(l)}, A^{(k)}, A^{(l)}, B^{(l)}) &\mapsto \\ \mapsto (p^{(k)}, \tau_k, \exp_{p^{(l)}} \tau_l \tilde{v}^{(l)}, \tau_l A^{(k)}, \tau_l A^{(l)} + B^{(l)}) &\equiv (p^{(k)}, \tau_k, \tilde{v}^{(k)}, A^{(k)}, B^{(k)}), \end{aligned}$$

where

$$\tilde{v}^{(l)}(p^{(l)}, \theta^{(l)}) = \left( v_\sigma^{(kl)}(p^{(l)}) + \sum_{i \neq \sigma} \theta_i^{(l)} v_i^{(kl)}(p^{(l)}) \right) / \sqrt{1 + \sum_{i \neq \sigma} (\theta_i^{(l)})^2}$$

for some  $\sigma$ . Note that  $Z_{kl}$  has normal crossings with the exceptional divisor  $E_k = \zeta_k^{-1}(Z_k) = \{\tau_k = 0\}$ , and that

$$W_{kl} \simeq f_{kl}(S_{kl}^+ \times (-1, 1))$$

up to a set of measure zero, where  $S_{kl}$  denotes <sup>2</sup> the sphere subbundle in  $\nu_{kl}$ , and we set  $S_{kl}^+ = \{v \in S_{kl} : v = \sum v_i v_i^{(kl)}, v_\sigma > 0\}$ . Taking into account that  $\exp_{p^{(l)}} \tau_l \tilde{v}^{(l)} = (\cos \tau_l) p^{(l)} + (\sin \tau_l) \tilde{v}^{(l)}$ , one sees that the phase function factorizes according to

$$\psi \circ (\text{id}_\xi \otimes (\zeta_k \circ \zeta_{kl})) = {}^{(kl)}\tilde{\psi}^{\text{tot}} = \tau_k \tau_l \cdot {}^{(kl)}\tilde{\psi}^{wk},$$

which in the given charts reads

$$\begin{aligned} \psi(x, \xi, X) &= \tau_k \left[ \left\langle \tau_l A^{(k)} p^{(k)} + (\tau_l A^{(l)} + B^{(l)}) \exp_{p^{(l)}} \tau_l \tilde{v}^{(l)}, \xi \right\rangle + \tau_k \tau_l \left\langle A^{(k)} \tilde{v}^{(k)}, \xi \right\rangle \right] \\ &= \tau_k \tau_l \left[ \left\langle A^{(k)} p^{(k)} + A^{(l)} p^{(l)} + B^{(l)} \tilde{v}^{(l)}, \xi \right\rangle + O(|\tau_k A^{(k)}|) + O(|\tau_l A^{(l)}|) + O(|\tau_l B^{(l)} \tilde{v}^{(l)}|) \right], \end{aligned}$$

<sup>2</sup>In order not to overload notation, we have denoted by  $S_{kl}$  and  $S_{k,l}$  two quite different sets.



where

$$\begin{aligned} O(|\tau_k A^{(k)}|) &= \tau_k \left\langle A^{(k)} \tilde{v}^{(k)}, \xi \right\rangle, \\ O(|\tau_l A^{(l)}|) &= \left\langle (\cos \tau_l - 1) A^{(l)} p^{(l)} + \sin \tau_l A^{(l)} \tilde{v}^{(l)}, \xi \right\rangle, \\ O(|\tau_l B^{(l)} \tilde{v}^{(l)}|) &= \left\langle (\tau_l^{-1} \sin \tau_l - 1) B^{(l)} \tilde{v}^{(l)}, \xi \right\rangle. \end{aligned}$$

Since

$$\begin{aligned} &\zeta_{kl}^{-1}(\{p^{(k)}\} \times \{\tau_k\} \times \pi_{kl}^{-1}(p^{(l)}) \times \mathfrak{g}_{p^{(k)}} \times \mathfrak{g}_{p^{(k)}}^\perp) \\ &= \{p^{(k)}\} \times \{\tau_k\} \times \{p^{(l)}\} \times (-T, T) \times (S_{kl}^+)_{p^{(l)}} \times \mathfrak{g}_{p^{(l)}} \times \mathfrak{g}_{p^{(l)}}^\perp \times \mathfrak{g}_{p^{(k)}}^\perp, \end{aligned}$$

we obtain for  $I_{kl}(\mu)$  the expression

$$\begin{aligned} I_{kl}(\mu) &= \int_{M_k(H_k) \times (-T, T)} \left[ \int_{(S_k)_{p^{(k)}, l}(H_l)} \left[ \int_{\zeta_{kl}^{-1}(\{p^{(k)}\} \times \{\tau_k\} \times \pi_{kl}^{-1}(p^{(l)}) \times \mathfrak{g}_{p^{(k)}} \times \mathfrak{g}_{p^{(k)}}^\perp) \times \mathbb{R}^n} e^{i \frac{\tau_k \tau_l}{\mu} (kl) \tilde{\psi}^{wk}} \right. \right. \\ &\quad \left. \left. \times ((\chi_k a \circ (\text{id}_\xi \otimes \zeta_k)) \chi_{kl}) \circ \zeta_{kl} \tilde{\Phi}_{kl} d\xi dA^{(k)} dA^{(l)} dB^{(l)} d\tilde{v}^{(l)} d\tau_l \right] dp^{(l)} \right] d\tau_k dp^{(k)} \\ &= \int_{M_k(H_k) \times (-T, T)} \left[ \int_{(S_k)_{p^{(k)}, l}(H_l) \times (-T, T)} \left[ \int_{(S_{kl}^+)_{p^{(l)}} \times \mathfrak{g}_{p^{(l)}} \times \mathfrak{g}_{p^{(l)}}^\perp \times \mathfrak{g}_{p^{(k)}}^\perp} \times \mathbb{R}^n} e^{i \frac{\tau_k \tau_l}{\mu} (kl) \tilde{\psi}^{wk}} \right. \right. \\ &\quad \left. \left. \times ((\chi_k a \circ (\text{id}_\xi \otimes \zeta_k)) \chi_{kl}) \circ \zeta_{kl} \tilde{\Phi}_{kl} d\xi dA^{(k)} dA^{(l)} dB^{(l)} d\tilde{v}^{(l)} \right] d\tau_l dp^{(l)} \right] d\tau_k dp^{(k)}, \end{aligned}$$

where  $d\tilde{v}^{(l)}$  is a suitable measure in  $(S_{kl}^+)_{p^{(l)}}$  such that we have the equality

$$dX dx \equiv \tilde{\Phi}_{kl} dA^{(k)} dA^{(l)} dB^{(l)} d\tilde{v}^{(l)} d\tau_l dp^{(l)} d\tau_k dp^{(k)}.$$

Furthermore,  $\tilde{\Phi}_{kl} = |\tau_l|^{c^{(l)} + d^{(k)} + d^{(l)} - 1} \Phi_{kl} \circ \zeta_{kl}$ .

**Second reduction.** Now, the group  $G_{p^{(k)}}$  acts on  $(S_k)_{p^{(k)}, l}$  with the isotropy types  $(H_l) = (H_{i_j}, (H_{i_{j+1}}), \dots, (H_L))$ . By the same arguments given in the first reduction, the isotropy types occurring in  $W_{kl}$  constitute a subset of these types, and we shall denote them by

$$(H_l) = (H_{i_{r_1}}, (H_{i_{r_2}}), \dots, (H_L)).$$

Consequently,  $G_{p^{(k)}}$  acts on  $S_{kl}$  with the isotropy types  $(H_{i_{r_2}}), \dots, (H_L)$ . Again, if  $G$  acted on  $S_{kl}$  only with type  $(H_L)$ , we shall see in the next section that the critical set of  $(kl) \tilde{\psi}^{wk}$  would be clean. However, in general this will not be the case, and we have to continue with the iteration.

**N-th decomposition.** The end of the iteration will be reached, once one arrives at a sphere bundle  $S_{klmn\dots}$  on which  $G$  acts only with the isotropy type  $(H_L)$ . More precisely, let  $(H_{i_1}), \dots, (H_{i_{N+1}}) = (H_L)$  be a branch of the isotropy tree of the  $G$ -action in  $\mathbb{R}^n$ ,  $N \geq 3$ , and consider for every fixed  $p^{(i_{N-1})} \in (S_{i_1 \dots i_{N-2}})_{p^{(i_{N-2}), i_{N-1}}}(H_{i_{N-1}})$  the decomposition of the closed  $G_{p^{(i_{N-1})}}$ -manifold  $(S_{i_1 \dots i_{N-1}})_{p^{(i_{N-1})}}$  given by

$$(S_{i_1 \dots i_{N-1}})_{p^{(i_{N-1})}} = W_{i_1 \dots i_N} \cup W_{i_1 \dots i_{N-1}L},$$

$$W_{i_1 \dots i_N} = f_{i_1 \dots i_N}(\overset{\circ}{D}_1(\nu_{i_1 \dots i_N})), \quad W_{i_1 \dots i_{N-1}L} = \text{Int}(S_{i_1 \dots i_{N-1}})_{p^{(i_{N-1}), L}},$$

where  $f_{i_1 \dots i_N} : \nu_{i_1 \dots i_N} \rightarrow (S_{i_1 \dots i_{N-1}})_{p^{(i_{N-1}), i_N}}$  is an invariant tubular neighborhood of the closed invariant submanifold  $(S_{i_1 \dots i_{N-1}})_{p^{(i_{N-1}), i_N}}(H_{i_N})$  in  $(S_{i_1 \dots i_{N-1}})_{p^{(i_{N-1}), i_N}} = (S_{i_1 \dots i_{N-1}})_{p^{(i_{N-1})}}$ , and

$$(S_{i_1 \dots i_{N-1}})_{p^{(i_{N-1}), L}} = (S_{i_1 \dots i_{N-1}})_{p^{(i_{N-1})}} - f_{i_1 \dots i_N}(\overset{\circ}{D}_{1/2}(\nu_{i_1 \dots i_N})).$$

Let  $\{\chi_{i_1 \dots i_N}, \chi_{i_1 \dots i_{N-1} L}\}$  denote a partition of unity subordinated to the covering by the open sets  $\{W_{i_1 \dots i_N}, \bar{W}_{i_1 \dots i_{N-1} L}\}$ , and decompose  $I_{i_1 \dots i_{N-1}}(\mu)$  accordingly, so that

$$I_{i_1 \dots i_{N-1}}(\mu) = I_{i_1 \dots i_N}(\mu) + I_{i_1 \dots i_{N-1} L}(\mu).$$

**N-th monoidal transformation.** In the chart  $(-T, T)^{N-1} \times S_{i_1 \dots i_{N-1}}^+ \times \mathfrak{g}$  consider the monoidal transformation

$$\zeta_{i_1 \dots i_N} : B_{Z_{i_1 \dots i_N}}((-T, T)^{N-1} \times S_{i_1 \dots i_{N-1}}^+ \times \mathfrak{g}) \longrightarrow (-T, T)^{N-1} \times S_{i_1 \dots i_{N-1}}^+ \times \mathfrak{g}$$

with center

$$\begin{aligned} Z_{i_1 \dots i_N} &= (-T, T)^{N-1} \times \mathbf{iso} S_{i_1 \dots i_{N-1}, i_N}^+(H_{i_N}), \\ S_{i_1 \dots i_{N-1}, i_N} &= \bigcup_{p^{(i_{N-1})}} (S_{i_1 \dots i_{N-1}})_{p^{(i_{N-1})}, i_N} = S_{i_1 \dots i_{N-1}}. \end{aligned}$$

The phase function then factorizes according to

$${}^{(i_1 \dots i_N)} \tilde{\psi}^{tot} = \tau_{i_1} \dots \tau_{i_N} {}^{(i_1 \dots i_N)} \tilde{\psi}^{wk},$$

where in the given charts

$${}^{(i_1 \dots i_N)} \tilde{\psi}^{wk} = \left\langle \sum_{j=1}^N A^{(i_j)} p^{(i_j)} + B^{(i_N)} \tilde{v}^{(i_N)}, \xi \right\rangle + \sum_{j=1}^N O(|\tau_{i_j} A^{(i_j)}|) + O(|\tau_{i_N} B^{(i_N)} \tilde{v}^{(i_N)}|),$$

and denoting <sup>3</sup> by  $S_{i_1 \dots i_N}$  the sphere bundle over  $(S_{i_1 \dots i_{N-1}})_{p^{(i_{N-1})}, i_N}(H_{i_N})$ , one finally obtains for the integral  $I_{i_1 \dots i_N}(\mu)$  the expression

$$\begin{aligned} I_{i_1 \dots i_N}(\mu) &= \int_{M_{i_1}(H_{i_1}) \times (-T, T)} \left[ \int_{(S_{i_1})_{p^{(i_1)}, i_2}(H_{i_2}) \times (-T, T)} \dots \left[ \int_{(S_{i_1 \dots i_{N-1}})_{p^{(i_{N-1})}, i_N}(H_{i_N}) \times (-T, T)} \right. \right. \\ (3) \quad &\left. \left. \int_{(S_{i_1 \dots i_N})_{p^{(i_N)} \times \mathfrak{g}_{p^{(i_N)}} \times \mathfrak{g}_{p^{(i_N)}}^+ \times \dots \times \mathfrak{g}_{p^{(i_1)}}^+ \times \mathbb{R}^n} e^{i \frac{\tau_1 \dots \tau_N}{\mu} {}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}} a_{i_1 \dots i_N} \tilde{\Phi}_{i_1 \dots i_N} \right. \right. \\ &\left. \left. d\xi dA^{(i_1)} \dots dA^{(i_N)} dB^{(i_N)} d\tilde{v}^{(i_N)} \right] d\tau_{i_N} dp^{(i_N)} \dots \right] d\tau_{i_2} dp^{(i_2)} \left] d\tau_{i_1} dp^{(i_1)}. \right. \end{aligned}$$

Here

$$a_{i_1 \dots i_N} = [a \chi_{i_1} \circ (\text{id } \xi \otimes \zeta_{i_1} \circ \zeta_{i_1 i_2} \circ \dots \circ \zeta_{i_1 \dots i_N})] [\chi_{i_1 i_2} \circ \zeta_{i_1 i_2} \circ \dots \circ \zeta_{i_1 \dots i_N}] \dots [\chi_{i_1 \dots i_N} \circ \zeta_{i_1 \dots i_N}]$$

is supposed to have compact support in one of the  $\theta^{(i_N)}$ -charts, and

$$\begin{aligned} \tilde{\Phi}_{i_1 \dots i_N} &= |\tau_{i_1}|^{c^{(i_1)} + d^{(i_1)} - 1} |\tau_{i_2}|^{c^{(i_2)} + d^{(i_1)} + d^{(i_2)} - 1} \dots |\tau_{i_N}|^{c^{(i_N)} + d^{(i_1)} + \dots + d^{(i_N)} - 1} \Phi_{i_1 \dots i_N} \\ &= \prod_{j=1}^N |\tau_{i_j}|^{c^{(i_j)} + \sum_r^j d^{(i_r)} - 1} \Phi_{i_1 \dots i_N}, \end{aligned}$$

where  $\Phi_{i_1 \dots i_N}$  is a smooth function which does not depend on the variables  $\tau_{i_j}$ .

**N-th reduction.** By assumption,  $G$  acts on  $S_{i_1 \dots i_N}$  only with type  $(H_L)$ , and the iteration process ends here.

<sup>3</sup>Again, note the different meaning of the notations  $S_{i_1 \dots i_N}$  and  $S_{i_1 \dots i_{N-1}, i_N}$ .

## 4. PHASE ANALYSIS OF THE WEAK TRANSFORM. THE FIRST FUNDAMENTAL THEOREM

We are now in position to state the first fundamental theorem in the derivation of equivariant spectral asymptotics. With the notation as in the previous section, consider an iteration of  $N$  steps along the branch  $((H_{i_1}), \dots, (H_{i_{N+1}}) = (H_L))$  of the isotropy tree of the  $G$ -action in  $\mathbb{R}^n$ , and let

$$\begin{aligned} p^{(i_1)} &\in M_{i_1}(H_{i_1}), \quad p^{(i_j)} \in (S_{i_1 \dots i_{j-1}}^+)_p^{(i_{j-1}), i_j}(H_{i_j}), \quad j = 2, \dots, N, \\ \mathfrak{g} &= \mathfrak{g}_{p^{(i_1)}} \oplus \mathfrak{g}_{p^{(i_1)}}^\perp = (\mathfrak{g}_{p^{(i_2)}} \oplus \mathfrak{g}_{p^{(i_2)}}^\perp) \oplus \mathfrak{g}_{p^{(i_1)}}^\perp = \dots = \mathfrak{g}_{p^{(i_N)}} \oplus \mathfrak{g}_{p^{(i_N)}}^\perp \oplus \dots \oplus \mathfrak{g}_{p^{(i_1)}}^\perp \\ d^{(i_j)} &= \dim \mathfrak{g}_{p^{(i_j)}}^\perp, \quad e^{(i_j)} = \dim \mathfrak{g}_{p^{(i_j)}}, \quad j = 1, \dots, N. \end{aligned}$$

As before,  $\{A_r^{(i_j)}(p^{(i_1)}, \dots, p^{(i_j)})\}$  will denote a basis of  $\mathfrak{g}_{p^{(i_j)}}^\perp$ , and  $\{B_r^{(i_N)}(p^{(i_1)}, \dots, p^{(i_N)})\}$  a basis of  $\mathfrak{g}_{p^{(i_N)}}$ . Let further

$$A^{(i_j)} = \sum_{r=1}^{d^{(i_j)}} \alpha_r^{(i_j)} A_r^{(i_j)}(p^{(i_1)}, \dots, p^{(i_j)}), \quad B^{(i_N)} = \sum_{r=1}^{e^{(i_N)}} \beta_r^{(i_N)} B_r^{(i_N)}(p^{(i_1)}, \dots, p^{(i_N)}),$$

and put

$$\tilde{v}^{(i_N)}(p^{(i_j)}, \theta^{(i_N)}) = \left( v_\varrho^{(i_1 \dots i_N)}(p^{(i_j)}) + \sum_{r \neq \varrho}^{e^{(i_N)}} \theta_r^{(i_N)} v_r^{(i_1 \dots i_N)}(p^{(i_j)}) \right) / \sqrt{1 + \sum_{r \neq \varrho} (\theta_r^{(i_N)})^2}$$

for some  $\varrho$ , where  $\{v_r^{(i_1 \dots i_N)}(p^{(i_1)}, \dots, p^{(i_N)})\}$  is an orthonormal frame in  $(\nu_{i_1 \dots i_N})_{p^{(i_N)}}$ . Finally, we shall use the notations

$$\begin{aligned} x^{(i_j \dots i_N)} &= \exp_{p^{(i_j)}}[\tau_{i_j} \exp_{p^{(i_{j+1})}}[\tau_{i_{j+1}} \exp_{p^{(i_{j+2})}}[\dots [\tau_{i_{N-2}} \exp_{p^{(i_{N-1})}}[\tau_{i_{N-1}} \exp_{p^{(i_N)}}[\tau_{i_N} \tilde{v}^{(i_N)}]]]] \dots]], \\ X^{(i_j \dots i_N)} &= \tau_{i_j} \dots \tau_{i_N} A^{(i_j)} + \tau_{i_{j+1}} \dots \tau_{i_N} A^{(i_{j+1})} + \dots + \tau_{i_{N-1}} \tau_{i_N} A^{(i_{N-1})} + \tau_{i_N} A^{(i_N)} + B^{(i_N)}, \end{aligned}$$

where  $j = 1, \dots, N$ . We then have the following

**Theorem 1.** *Consider the factorization*

$${}^{(i_1 \dots i_N)} \tilde{\psi}^{tot} = \psi(x^{(i_1 \dots i_N)}, \xi, X^{(i_1 \dots i_N)}) = \tau_{i_1} \dots \tau_{i_N} {}^{(i_1 \dots i_N)} \tilde{\psi}^{wk, pre}$$

of the phase function  $\psi$  after  $N$  iteration steps, where

$${}^{(i_1 \dots i_N)} \tilde{\psi}^{wk, pre} = \left\langle \sum_{j=1}^N A^{(i_j)} p^{(i_j)} + B^{(i_N)} \tilde{v}^{(i_N)}, \xi \right\rangle + \sum_{j=1}^N O(|\tau_{i_j} A^{(i_j)}|) + O(|\tau_{i_N} B^{(i_N)} \tilde{v}^{(i_N)}|).$$

By construction, for  $\tau_{i_j} \neq 0$ ,  $1 \leq j \leq N$ , the  $G$ -orbit through  $x^{(i_1 \dots i_N)}$  is of principal type  $G/H_L$ , which is equivalent to say that  $G$  acts on  $S_{i_1 \dots i_N}$  only with the isotropy type  $(H_L)$ . Let further

$${}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}$$

denote the pullback of  ${}^{(i_1 \dots i_N)} \tilde{\psi}^{wk, pre}$  along the substitution  $\tau = \delta_{i_1 \dots i_N}(\sigma)$  given by the sequence of monoidal transformations

$$\begin{aligned} \delta_{i_1 \dots i_N} : (\sigma_{i_1}, \dots, \sigma_{i_N}) &\mapsto \sigma_{i_1}(1, \sigma_{i_2}, \dots, \sigma_{i_N}) = (\sigma'_{i_1}, \dots, \sigma'_{i_N}) \mapsto \sigma'_{i_2}(\sigma'_{i_1}, 1, \dots, \sigma'_{i_N}) = (\sigma''_{i_1}, \dots, \sigma''_{i_N}) \\ &\mapsto \sigma''_{i_3}(\sigma''_{i_1}, \sigma''_{i_2}, 1, \dots, \sigma''_{i_N}) = \dots \mapsto \dots = (\tau_{i_1}, \dots, \tau_{i_N}). \end{aligned}$$

Then the critical set  $\text{Crit}({}^{(i_1 \dots i_N)} \tilde{\psi}^{wk})$  of  ${}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}$  is given by all points with coordinates

$$(\sigma_{i_1}, \dots, \sigma_{i_N}, p^{(i_1)}, \dots, p^{(i_N)}, \tilde{v}^{(i_N)}, \alpha^{(i_1)}, \dots, \alpha^{(i_N)}, \beta^{(i_N)}, \xi)$$

satisfying the conditions

- (I)  $\alpha^{(i_j)} = 0$  for all  $j = 1, \dots, N$ , and  $\sum \beta_r^{(i_N)} B_r^{(i_N)} \tilde{v}^{(i_N)} = 0$ ;
- (II)  $\xi \perp (\mathfrak{g}_{p^{(i_1)}}^\perp \cdot x^{(i_1 \dots i_N)}); \quad \xi \perp (\mathfrak{g}_{p^{(i_2)}}^\perp \cdot x^{(i_2 \dots i_N)}); \dots \quad \xi \perp (\mathfrak{g}_{p^{(i_N)}}^\perp \cdot x^{(i_N)});$
- (III)  $\xi \perp (\mathfrak{g}_{p^{(i_N)}} \cdot \tilde{v}^{(i_N)})$ .

Furthermore,  $\text{Crit}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}$  is a  $C^\infty$ -submanifold of codimension  $2\kappa$ , where  $\kappa = \dim G/H_L$  is the dimension of a principal orbit.

*Proof.* Let us compute first the derivatives with respect to  $\xi$ , and assume that all  $\sigma_{i_j}$  are different from zero. Then all  $\tau_{i_j}$  are different from zero, too, and  $\partial_\xi^{(i_1 \dots i_N)} \tilde{\psi}^{wk} = 0$  is equivalent to

$$\frac{1}{\tau_{i_1} \dots \tau_{i_N}} \partial_\xi \psi(x^{(i_1 \dots i_N)}, \xi, X^{(i_1 \dots i_N)}) = 0,$$

which gives us the condition  $X^{(i_1 \dots i_N)} \in \mathfrak{g}_{x^{(i_1 \dots i_N)}}$ . Since for sufficiently small  $\tau_{i_j}$  the point  $x^{(i_1 \dots i_N)}$  lies in a slice in  $N_{p^{(i_1)}}(G \cdot p^{(i_1)})$ , the element  $X^{(i_1 \dots i_N)}$  must annihilate  $p^{(i_1)}$  as well. But

$$\mathfrak{g}_{p^{(i_N)}} \subset \mathfrak{g}_{p^{(i_{N-1})}} \subset \dots \subset \mathfrak{g}_{p^{(i_1)}}$$

and  $\mathfrak{g}_{p^{(i_{j+1})}}^\perp \subset \mathfrak{g}_{p^{(i_j)}}$  imply

$$X^{(i_1 \dots i_N)} p^{(i_1)} = \tau_{i_1} \dots \tau_{i_N} \sum \alpha_r^{(i_1)} A_r^{(i_1)} p^{(i_1)} = 0.$$

Thus we conclude  $\alpha^{(i_1)} = 0$ , which gives  $X^{(i_2 \dots i_N)} \in \mathfrak{g}_{x^{(i_2 \dots i_N)}}$ , and consequently  $X^{(i_2 \dots i_N)} \in \mathfrak{g}_{x^{(i_2 \dots i_N)}}$ . Repeating the above argument we actually obtain

$$(4) \quad \mathfrak{g}_{x^{(i_1 \dots i_N)}} = \mathfrak{g}_{\tilde{v}^{(i_N)}},$$

since  $\mathfrak{g}_{\tilde{v}^{(i_N)}} \subset \mathfrak{g}_{p^{(i_N)}}$ , and therefore condition I) in the case that all  $\sigma_{i_j}$  are different from zero. Let now one of the  $\sigma_{i_j}$  be equal to zero. Then all  $\tau_{i_j}$  are zero, too, and  $\partial_\xi^{(i_1 \dots i_N)} \tilde{\psi}^{wk} = 0$  is equivalent to

$$(5) \quad \sum_{j=1}^N \left( \sum_r \alpha_r^{(i_j)} A_r^{(i_j)} \right) p^{(i_j)} + \sum_r \beta_r^{(i_N)} B_r^{(i_N)} \tilde{v}^{(i_N)} = 0.$$

Now, for every  $j = 1, \dots, N$ , the group  $G_{p^{(i_j)}}$  acts orthogonally on the space  $N_{p^{(i_j)}}(G_{p^{(i_{j-1})}} \cdot p^{(i_j)})$ , where we understand that  $G_{p^{(i_0)}} = G$ . Furthermore, by construction we have

$$N_{p^{(i_{j+1})}}(G_{p^{(i_j)}} \cdot p^{(i_{j+1})}) \subset N_{p^{(i_j)}}(G_{p^{(i_{j-1})}} \cdot p^{(i_j)}),$$

so that

$$(6) \quad V^{(i_1 \dots i_j)} = \bigcap_{r=1}^j N_{p^{(i_r)}}(G_{p^{(i_{r-1})}} \cdot p^{(i_r)}) = N_{p^{(i_j)}}(G_{p^{(i_{j-1})}} \cdot p^{(i_j)}).$$

Since  $p^{(i_j)} \in (S_{i_1 \dots i_{j-1}})_{p^{(i_{j-1})}} \subset V^{(i_1 \dots i_{j-1})}$ , we therefore see that for every  $j = 2, \dots, N$

$$\sum_r \alpha_r^{(i_j)} A_r^{(i_j)} p^{(i_j)} \in \mathfrak{g}_{p^{(i_j)}}^\perp \cdot p^{(i_j)} = T_{p^{(i_j)}}(G_{p^{(i_{j-1})}} \cdot p^{(i_j)}) \subset V^{(i_1 \dots i_{j-1})}.$$

In addition, one has of course  $\sum_r \alpha_r^{(i_1)} A_r^{(i_1)} p^{(i_1)} \in \mathfrak{g}_{p^{(i_1)}}^\perp \cdot p^{(i_1)} = T_{p^{(i_1)}}(G \cdot p^{(i_1)})$ , as well as

$$\sum_r \beta_r^{(i_N)} B_r^{(i_N)} \tilde{v}^{(i_N)} \in V^{(i_1 \dots i_N)},$$

so that taking everything together we obtain

$$(7) \quad \partial_\xi^{(i_1 \dots i_N)} \tilde{\psi}^{wk} = 0 \iff \alpha^{(i_j)} = 0 \quad \forall \quad j = 1, \dots, N \quad \text{and} \quad \sum_r \beta_r^{(i_N)} B_r^{(i_N)} \tilde{v}^{(i_N)} = 0.$$

Let us consider next the  $\alpha$ -derivatives. Clearly,

$$\begin{aligned}\partial_{\alpha^{(i_1)}} (i_1 \dots i_N) \tilde{\psi}^{wk} = 0 &\iff \langle Y x^{(i_1 \dots i_N)}, \xi \rangle = 0 \quad \forall Y \in \mathfrak{g}_{p^{(i_1)}}^\perp, \\ \partial_{\alpha^{(i_2)}} (i_1 \dots i_N) \tilde{\psi}^{wk} = 0 &\iff \langle Y x^{(i_2 \dots i_N)}, \xi \rangle = 0 \quad \forall Y \in \mathfrak{g}_{p^{(i_2)}}^\perp.\end{aligned}$$

Now, assuming for a moment that all the  $\sigma_{i_j}$  are different from zero, one computes for the remaining derivatives that

$$\begin{aligned}\partial_{\alpha_r^{(i_j)}} (i_1 \dots i_N) \tilde{\psi}^{wk} &= \frac{1}{\tau_{i_1} \dots \tau_{i_N}} \langle \tau_{i_j} \dots \tau_{i_N} A_r^{(i_j)} x^{(i_1 \dots i_N)}, \xi \rangle \\ &= \frac{1}{\tau_{i_1} \dots \tau_{i_{j-1}}} \langle A_r^{(i_j)} (\tau_{i_1} \sin \tau_{i_2} \dots \sin \tau_{i_{j-1}}) x^{(i_j \dots i_N)}, \xi \rangle,\end{aligned}$$

since  $A_r^{(i_j)} \in \mathfrak{g}_{p^{(i_j)}}^\perp \subset \mathfrak{g}_{p^{(i_{j-1})}} \subset \dots \subset \mathfrak{g}_{p^{(i_1)}}$ . From this one deduces for arbitrary  $\sigma_{i_j}$  that for  $j = 1, \dots, N$

$$(8) \quad \partial_{\alpha^{(i_j)}} (i_1 \dots i_N) \tilde{\psi}^{wk} = 0 \iff \langle Y x^{(i_j \dots i_N)}, \xi \rangle = 0 \quad \forall Y \in \mathfrak{g}_{p^{(i_j)}}^\perp.$$

In a similar way, it is not difficult to see that

$$(9) \quad \partial_{\beta^{(i_N)}} (i_1 \dots i_N) \tilde{\psi}^{wk} = 0 \iff \langle Z \tilde{v}^{(i_N)}, \xi \rangle = 0 \quad \forall Z \in \mathfrak{g}_{p^{(i_N)}}.$$

by which the necessity of the conditions (I)–(III) is established. In order to see their sufficiency, let them be fulfilled, and let us assume again that  $\sigma_{i_j} \neq 0$  for all  $j = 1, \dots, N$ . Then (II) and (III) imply that

$$\langle Z \exp_{p^{(i_N)}} \tau_{i_N} \tilde{v}^{(i_N)}, \xi \rangle = 0 \quad \forall Z \in \mathfrak{g}_{p^{(i_{N-1})}},$$

since  $\mathfrak{g}_{p^{(i_{N-1})}} = \mathfrak{g}_{p^{(i_N)}} \oplus \mathfrak{g}_{p^{(i_N)}}^\perp$ . By repeatedly using (II) we therefore conclude

$$(10) \quad \xi \in N_{x^{(i_1 \dots i_N)}}(G \cdot x^{(i_1 \dots i_N)}).$$

Now, by construction,  $G \cdot x^{(i_1 \dots i_N)}$  is of principal type  $G/H_L$  in  $\mathbb{R}^n$ , so that the isotropy group of  $x^{(i_1 \dots i_N)}$  must act trivially on  $N_{x^{(i_1 \dots i_N)}}(G \cdot x^{(i_1 \dots i_N)})$ , compare Bredon [3], page 181. In addition, by (I) and Equation (4),  $\sum_r \beta_r^{(i_N)} B_r^{(i_N)} \in \mathfrak{g}_{\tilde{v}^{(i_N)}} = \mathfrak{g}_{x^{(i_1 \dots i_N)}}$ . The relation (10) therefore implies  $\sum_r \beta_r^{(i_N)} B_r^{(i_N)} \xi = 0$ . Let us consider now the case where at least one of the  $\sigma_{i_j}$  equals zero, so that all  $\tau_{i_j} = 0$ . Then (II) means that  $\xi \in V^{(i_1 \dots i_N)}$ . We shall now need the following simple

**Lemma 1.** *The orbit of the point  $\tilde{v}^{(i_N)}$  in the  $G_{p^{(i_N)}}$ -space  $V^{(i_1 \dots i_N)}$  is of principal type.*

*Proof of the lemma.* By assumption, for  $\sigma_{i_j} \neq 0$ ,  $1 \leq j \leq N$ , the  $G$ -orbit of  $x^{(i_1 \dots i_N)}$  is of principal type  $G/H_L$  in  $\mathbb{R}^n$ . The theory of compact group actions then implies that this is equivalent to the fact that  $x^{(i_2 \dots i_N)} \in V^{(i_1)}$  is of principal type in the  $G_{p^{(i_1)}}$ -space  $V^{(i_1)}$ , see Bredon [3], page 181, which in turn is equivalent to the fact that  $x^{(i_3 \dots i_N)} \in V^{(i_1 i_2)}$  is of principal type in the  $G_{p^{(i_2)}}$ -space  $V^{(i_1 i_2)}$ , and so forth. Thus,  $x^{(i_j \dots i_N)} \in V^{(i_1 \dots i_{j-1})}$  must be of principal type in the  $G_{p^{(i_{j-1})}}$ -space  $V^{(i_1 \dots i_{j-1})}$  for all  $j = 1, \dots, N$ , and the assertion follows.  $\square$

Now (III) implies that  $\xi \in V^{(i_1 \dots i_N)} \cap N_{\tilde{v}^{(i_N)}}(G_{p^{(i_N)}} \cdot \tilde{v}^{(i_N)})$ . But the previous lemma implies that  $G_{\tilde{v}^{(i_N)}}$  acts trivially on the latter space, so that by condition (I) we obtain again the condition  $\sum_r \beta_r^{(i_N)} B_r^{(i_N)} \xi = 0$ . Collecting everything together we finally obtain

$$(11) \quad \text{(I), (II), (III)} \implies \sum_r \beta_r^{(i_N)} B_r^{(i_N)} \xi = 0.$$

Now, by (7) – (9) we have

$$(I), (II), (III) \iff \partial_{\xi, \alpha^{(i_1)}, \dots, \alpha^{(i_N)}, \beta^{(i_N)}} (i_1 \dots i_N) \tilde{\psi}^{wk} = 0.$$

It therefore remains to study the derivatives with respect to the variables  $\sigma_{i_j}$ ,  $p^{(i_j)}$ , and  $\tilde{v}^{(i_N)}$ . It is immediately clear that

$$(I) \implies \partial_{\sigma} (i_1 \dots i_N) \tilde{\psi}^{wk} = 0.$$

By (11) we further have

$$(I), (II), (III) \implies \partial_{\tilde{v}^{(i_N)}} (i_1 \dots i_N) \tilde{\psi}^{wk} = 0.$$

Let us now take for each  $j = 1, \dots, N$  local coordinates  $p^{(i_j)} = p^{(i_j)}(s^{(i_j)}) \in (S_{i_1 \dots i_{j-1}}^+)_p^{(i_{j-1}), i_j}(H_{i_j})$  around  $p_0^{(i_j)} = p^{(i_j)}(0)$ , and write

$$B^{(i_N)} = g(p^{(i_N)}) B_0^{(i_N)} g^{-1}(p^{(i_N)}) \in \mathfrak{g}_{p^{(i_N)}}, \quad B_0^{(i_N)} \in \mathfrak{g}_{p_0^{(i_N)}}, \quad g(p^{(i_N)}) \in G.$$

On then computes

$$\frac{\partial B^{(i_N)}}{\partial s_r^{(i_N)}} = \left( \frac{\partial g(p^{(i_N)}(s^{(i_N)}))}{\partial s_r^{(i_N)}} \right) g(p^{(i_N)}(s^{(i_N)}))^{-1} B^{(i_N)} + B^{(i_N)} g(p^{(i_N)}(s^{(i_N)})) \left( \frac{\partial g^{-1}(p^{(i_N)}(s^{(i_N)}))}{\partial s_r^{(i_N)}} \right),$$

so that with (11) one finally concludes

$$(I), (II), (III) \implies \partial_{p^{(i_1)}, \dots, p^{(i_N)}} (i_1 \dots i_N) \tilde{\psi}^{wk} = 0.$$

Thus we have computed the critical set of  $(i_1 \dots i_N) \tilde{\psi}^{wk}$ , and it remains to show that it is a  $C^\infty$ -submanifold of codimension  $2\kappa$ . For this end, let us define the subspaces

$$(12) \quad E^{(i_j)} = \mathfrak{g}_{p^{(i_j)}}^\perp \cdot x^{(i_j \dots i_N)}, \quad F^{(i_N)} = \mathfrak{g}_{p^{(i_N)}} \cdot \tilde{v}^{(i_N)}.$$

One has  $E^{(i_1)} \subset \mathfrak{g} \cdot x^{(i_1 \dots i_N)} = T_{x^{(i_1 \dots i_N)}}(G \cdot x^{(i_1 \dots i_N)})$ , as well as

$$E^{(i_j)} \subset \mathfrak{g}_{p^{(i_{j-1})}} \cdot x^{(i_j \dots i_N)} = T_{x^{(i_j \dots i_N)}}(G_{p^{(i_{j-1})}} \cdot x^{(i_j \dots i_N)}) \subset V^{(i_1 \dots i_{j-1})}$$

for  $2 \leq j \leq N$ . Similarly,  $F^{(i_N)} \subset V^{(i_1 \dots i_N)}$ . Now, for small  $\tau_{i_j}$ , we clearly have  $E^{(i_j)} \cap V^{(i_1 \dots i_j)} = \{0\}$ , so that we obtain the direct sum of vector spaces

$$E^{(i_1)} \oplus E^{(i_2)} \oplus \dots \oplus E^{(i_N)} \oplus F^{(i_N)}.$$

We therefore arrive at the characterization

$$(13) \quad \text{Crit}((i_1 \dots i_N) \tilde{\psi}^{wk}) = \left\{ \alpha^{(i_j)} = 0, \quad \sum_r \beta_r^{(i_N)} B_r^{(i_N)} \tilde{v}^{(i_N)} = 0, \quad \xi \perp \left( \bigoplus_{j=1}^N E^{(i_j)} \oplus F^{(i_N)} \right) \right\},$$

Note that the condition  $\sum_r \beta_r^{(i_N)} B_r^{(i_N)} \xi = 0$  is already implied by the others. Now, for small, but arbitrary  $\sigma_{i_j}$  one has

$$\dim E^{(i_j)} = \dim \mathfrak{g}_{p^{(i_j)}}^\perp \cdot p^{(i_j)} = \dim G_{p^{(i_{j-1})}} \cdot p^{(i_j)}.$$

Since for  $\sigma_{i_j} \neq 0$ ,  $1 \leq j \leq N$ , the  $G$ -orbit of  $x^{(i_1 \dots i_N)}$  is of principal type  $G/H_L$  in  $\mathbb{R}^n$  by assumption, one computes in this case

$$\begin{aligned} \kappa &= \dim G \cdot x^{(i_1 \dots i_N)} = \dim \mathfrak{g} \cdot x^{(i_1 \dots i_N)} = \dim[\mathfrak{g}_{p^{(i_N)}} \oplus \mathfrak{g}_{p^{(i_N)}}^\perp \oplus \dots \oplus \mathfrak{g}_{p^{(i_1)}}^\perp] \cdot x^{(i_1 \dots i_N)} \\ &= \dim[\mathfrak{g}_{p^{(i_1)}}^\perp \cdot x^{(i_1 \dots i_N)} + \tau_{i_1} \mathfrak{g}_{p^{(i_2)}}^\perp \cdot x^{(i_2 \dots i_N)} + \tau_{i_1} \sin \tau_{i_2} \mathfrak{g}_{p^{(i_3)}}^\perp \cdot x^{(i_3 \dots i_N)} + \dots \\ &\quad + \tau_{i_1} \sin \tau_{i_2} \dots \sin \tau_{i_{N-1}} \mathfrak{g}_{p^{(i_N)}}^\perp \cdot x^{(i_N)} + \tau_{i_1} \sin \tau_{i_2} \dots \sin \tau_{i_N} \mathfrak{g}_{p^{(i_N)}} \tilde{v}^{(i_N)}] \\ &= \dim[E^{(i_1)} \oplus \tau_{i_1} E^{(i_2)} \oplus \tau_{i_1} \sin \tau_{i_2} E^{(i_3)} \oplus \dots \oplus \tau_{i_1} \sin \tau_{i_2} \dots \sin \tau_{i_{N-1}} E^{(i_N)} \\ &\quad \oplus \tau_{i_1} \sin \tau_{i_2} \dots \sin \tau_{i_N} F^{(i_N)}] = \sum_{j=1}^N \dim E^{(i_j)} + \dim F^{(i_N)}. \end{aligned}$$

But since the dimension of the spaces  $E^{(i_j)}$  and  $F^{(i_N)}$  does not depend on the variables  $\sigma_{i_j}$ , we obtain for sufficiently small, but arbitrary  $\sigma_{i_j}$  the equality

$$(14) \quad \kappa = \sum_{j=1}^N \dim E^{(i_j)} + \dim F^{(i_N)}.$$

Note that, in contrast, the dimension of  $\mathfrak{g} \cdot x^{(i_1 \dots i_N)}$  collapses, as soon as one of the  $\tau_{i_j}$  becomes zero. Thus we arrive at a vector bundle with  $(n - \kappa)$ -dimensional fiber that is locally given by the trivialization

$$(\sigma_{i_1}, \dots, \sigma_{i_N}, p^{(i_1)}, \dots, p^{(i_N)}, \tilde{v}^{(i_N)}, (\bigoplus_{j=1}^N E^{(i_j)} \oplus F^{(i_N)})^\perp) \mapsto (\sigma_{i_1}, \dots, \sigma_{i_N}, p^{(i_1)}, \dots, p^{(i_N)}, \tilde{v}^{(i_N)}).$$

Consequently, by Equation (13) we see that  $\text{Crit}((i_1 \dots i_N) \tilde{\psi}^{wk})$  is equal to the fiber product of the mentioned vector bundle with the isotropy algebra bundle that is given by the local trivialization

$$(\sigma_{i_1}, \dots, \sigma_{i_N}, p^{(i_1)}, \dots, p^{(i_N)}, \tilde{v}^{(i_N)}, \mathfrak{g}_{\tilde{v}^{(i_N)}}) \mapsto (\sigma_{i_1}, \dots, \sigma_{i_N}, p^{(i_1)}, \dots, p^{(i_N)}, \tilde{v}^{(i_N)}).$$

Lastly, since by Equation (4) we have  $\mathfrak{g}_{\tilde{v}^{(i_N)}} = \mathfrak{g}_{x^{(i_1, \dots, i_N)}}$  in case that all  $\sigma_{i_j}$  are different from zero, we necessarily have  $\dim \mathfrak{g}_{\tilde{v}^{(i_N)}} = d - \kappa$ , which concludes the proof of the theorem.  $\square$

## 5. PHASE ANALYSIS OF THE WEAK TRANSFORM. THE SECOND FUNDAMENTAL THEOREM

In this section, we shall prove the second fundamental theorem in the derivation of equivariant spectral asymptotics for orthogonal compact group actions. The notation will be the same as in the previous sections.

**Theorem 2.** *Let*

$$(i_1 \dots i_N) \tilde{\psi}^{tot} = \tau_{i_1} \dots \tau_{i_N} (i_1 \dots i_N) \tilde{\psi}^{wk, pre} = \tau_{i_1}(\sigma) \dots \tau_{i_N}(\sigma) (i_1 \dots i_N) \tilde{\psi}^{wk}$$

*denote the factorization of the phase function after  $N$  iteration steps along the isotropy branch  $((H_{i_1}), \dots, (H_{i_{N+1}}) = (H_L))$ . By construction, for  $\tau_{i_j} \neq 0$ ,  $1 \leq j \leq N$ , the  $G$ -orbit through  $x^{(i_1 \dots i_N)}$  is of principal type  $G/H_L$ . Then for each point of the critical manifold  $\text{Crit}((i_1 \dots i_N) \tilde{\psi}^{wk})$ , the restriction of*

$$\text{Hess}^{(i_1 \dots i_N) \tilde{\psi}^{wk}}$$

*to the normal space to  $\text{Crit}((i_1 \dots i_N) \tilde{\psi}^{wk})$  at the given point defines a non-degenerate symmetric bilinear form.*

Before proving the theorem, let us make the following general observation. Let  $M$  be a  $n$ -dimensional Riemannian manifold, and  $C$  the critical set of a function  $\psi \in C^\infty(M)$ , which is assumed to be a smooth submanifold in a chart  $\mathcal{O} \subset M$ . Let further

$$\alpha : (x, y) \mapsto p, \quad \beta : (m_1, \dots, m_n) \mapsto p, \quad p \in \mathcal{O},$$

be two systems of local coordinates on  $\mathcal{O}$ , such that  $\alpha(x, y) \in C$  if and only if  $y = 0$ . One computes

$$\partial_{y_l}(\psi \circ \alpha)(x, y) = \sum_{i=1}^n \frac{\partial(\psi \circ \beta)}{\partial m_i}(\beta^{-1} \circ \alpha(x, y)) \partial_{y_l}(\beta^{-1} \circ \alpha)_i(x, y),$$

as well as

$$\begin{aligned} \partial_{y_k} \partial_{y_l}(\psi \circ \alpha)(x, y) &= \sum_{i=1}^n \frac{\partial(\psi \circ \beta)}{\partial m_i}(\beta^{-1} \circ \alpha(x, y)) \partial_{y_k} \partial_{y_l}(\beta^{-1} \circ \alpha)_i(x, y) \\ &+ \sum_{i,j=1}^n \frac{\partial^2(\psi \circ \beta)}{\partial m_i \partial m_j}(\beta^{-1} \circ \alpha(x, y)) \partial_{y_k}(\beta^{-1} \circ \alpha)_j(x, y) \partial_{y_l}(\beta^{-1} \circ \alpha)_i(x, y). \end{aligned}$$

Since

$$\alpha_{*,(x,y)}(\partial_{y_k}) = \sum_{j=1}^n \partial_{y_k}(\beta^{-1} \circ \alpha)_j(x, y) \beta_{*,(\beta^{-1} \circ \alpha)(x,y)}(\partial_{m_j}),$$

this implies

$$(15) \quad \partial_{y_k} \partial_{y_l}(\psi \circ \alpha)(x, 0) = \text{Hess } \psi|_{\alpha(x,0)}(\alpha_{*,(x,0)}(\partial_{y_k}), \alpha_{*,(x,0)}(\partial_{y_l})),$$

by definition of the Hessian. Let us now write  $x = (x', x'')$ , and consider the restriction of  $\psi$  onto the  $C^\infty$ -submanifold

$$M_{c'} = \{m \in \mathcal{O} : m = \alpha(c', x'', y)\}.$$

We write  $\psi_{c'} = \psi|_{M_{c'}}$ , and denote the critical set of  $\psi_{c'}$  by  $C_{c'}$ , which contains  $C \cap M_{c'}$  as a subset. Introducing on  $M_{c'}$  the local coordinates

$$\alpha' : (x'', y) \mapsto \alpha(c', x'', y),$$

we obtain

$$(16) \quad \partial_{y_k} \partial_{y_l}(\psi_{c'} \circ \alpha')(x'', 0) = \text{Hess } \psi_{c'}|_{\alpha'(x'',0)}(\alpha'_{*,(x'',0)}(\partial_{y_k}), \alpha'_{*,(x'',0)}(\partial_{y_l})).$$

Let us now assume  $C_{c'} = C \cap M_{c'}$ , a transversal intersection. Then  $C_{c'}$  is a submanifold of  $M_{c'}$ , and the normal space to  $C_{c'}$  as a submanifold of  $M_{c'}$  at a point  $\alpha'(x'', 0)$  is spanned by the vector fields  $\alpha'_{*,(x'',0)}(\partial_{y_k})$ . Since clearly

$$\partial_{y_k} \partial_{y_l}(\psi_{c'} \circ \alpha')(x'', 0) = \partial_{y_k} \partial_{y_l}(\psi \circ \alpha)(x, 0), \quad x = (c', x''),$$

we thus have proven the following

**Lemma 2.** *Assume that  $C_{c'} = C \cap M_{c'}$ . Then the restriction*

$$\text{Hess } \psi(\alpha(c', x'', 0))|_{N_{\alpha(c', x'', 0)}C}$$

*of the Hessian of  $\psi$  to the normal space  $N_{\alpha(c', x'', 0)}C$  defines a non-degenerate quadratic form if, and only if the restriction*

$$\text{Hess } \psi_{c'}(\alpha'(x'', 0))|_{N_{\alpha'(x'', 0)}C_{c'}}$$

*of the Hessian of  $\psi_{c'}$  to the normal space  $N_{\alpha'(x'', 0)}C_{c'}$  defines a non-degenerate quadratic form.*

□



*Proof of second fundamental theorem.* Let us begin by noting that with respect to the standard coordinates in  $\mathbb{R}^n$  and  $\mathfrak{g}$ , the Hessian of  $\psi$  is given by the matrix

$$\begin{pmatrix} 0 & \langle X e_i, e_j \rangle & -\langle e_i, X_j \xi \rangle \\ \langle X e_j, e_i \rangle & 0 & \langle X_j x, e_i \rangle \\ -\langle e_j, X_i \xi \rangle & \langle X_i x, e_j \rangle & 0 \end{pmatrix},$$

where  $\{e_j\}$  and  $\{X_j\}$  denote the standard basis in  $\mathbb{R}^n$  and  $\mathfrak{g}$ , respectively. A direct computation then shows that its restriction to the normal space  $N_{(x,\xi,X)} \text{Reg Crit}(\psi)$  defines a non-degenerate quadratic form for all  $(x, \xi, X) \in \text{Reg Crit}(\psi)$ . Now, for  $\sigma_{i_1} \cdots \sigma_{i_N} \neq 0$ , the sequence of monoidal transformations  $\zeta_{i_1} \circ \zeta_{i_1 i_2} \circ \cdots \circ \zeta_{i_1 \dots i_N}$ , composed with the transformation  $\delta_{i_1 \dots i_N}$ , constitutes a diffeomorphism, so that in the given charts of the resolution the restriction of

$$\text{Hess}^{(i_1 \dots i_N)} \tilde{\psi}^{tot}(\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}, \alpha^{(i_j)}, \beta^{(i_N)}, \xi)$$

to the normal space of the total transform

$$\tilde{\mathcal{C}}^{tot} = ((\zeta_{i_1} \circ \zeta_{i_1 i_2} \circ \cdots \circ \zeta_{i_1 \dots i_N} \circ \delta_{i_1 \dots i_N}) \otimes \text{id}_\xi)^{-1}(\text{Crit}(\psi))$$

defines a non-degenerate quadratic form as well at every point with  $\sigma_{i_1} \cdots \sigma_{i_N} \neq 0$ . Next, one computes

$$\begin{aligned} \left( \frac{\partial^2 (i_1 \dots i_N) \tilde{\psi}^{tot}}{\partial \gamma_k \partial \gamma_l} \right)_{k,l} &= \tau_{i_1}(\sigma) \cdots \tau_{i_N}(\sigma) \left( \frac{\partial^2 (i_1 \dots i_N) \tilde{\psi}^{wk}}{\partial \gamma_k \partial \gamma_l} \right)_{k,l} \\ &+ \begin{pmatrix} \left( \frac{\partial^2 (\tau_{i_1}(\sigma) \cdots \tau_{i_N}(\sigma))}{\partial \sigma_{i_r} \sigma_{i_s}} \right)_{r,s} & 0 \\ 0 & 0 \end{pmatrix} (i_1 \dots i_N) \tilde{\psi}^{wk} + R \end{aligned}$$

where  $R$  represents a matrix whose entries contain first order derivatives of  $(i_1 \dots i_N) \tilde{\psi}^{wk}$  as factors. But since

$$\tilde{\mathcal{C}}_{|\sigma_{i_1} \cdots \sigma_{i_N} \neq 0}^{tot} = \text{Crit}((i_1 \dots i_N) \tilde{\psi}^{wk})_{|\sigma_{i_1} \cdots \sigma_{i_N} \neq 0}$$

because  $(i_1 \dots i_N) \tilde{\psi}^{tot}$  and  $(i_1 \dots i_N) \tilde{\psi}^{wk}$  vanish on their critical sets, we conclude that the transversal Hessian of  $(i_1 \dots i_N) \tilde{\psi}^{wk}$  does not degenerate along  $\text{Crit}((i_1 \dots i_N) \tilde{\psi}^{wk})_{|\sigma_{i_1} \cdots \sigma_{i_N} \neq 0}$ . Therefore, it remains to study the transversal Hessian of  $(i_1 \dots i_N) \tilde{\psi}^{wk}$  in the case that any of the  $\sigma_{i_j}$  vanishes. Now, the proof of the first fundamental theorem showed that

$$\partial_{\xi, \alpha^{(i_1)}, \dots, \alpha^{(i_N)}, \beta^{(i_N)}} (i_1 \dots i_N) \tilde{\psi}^{wk} = 0 \implies \partial_{\sigma_{i_1}, \dots, \sigma_{i_N}, p^{(i_1)}, \dots, p^{(i_N)}, \tilde{v}^{(i_N)}} (i_1 \dots i_N) \tilde{\psi}^{wk} = 0.$$

If therefore

$$(i_1 \dots i_N) \tilde{\psi}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}^{wk}(\alpha^{(i_j)}, \beta^{(i_N)}, \xi)$$

denotes the weak transform of the phase function  $\psi$  regarded as a function of the variables  $(\alpha^{(i_1)}, \dots, \alpha^{(i_N)}, \beta^{(i_N)}, \xi)$  alone, while the variables  $(\sigma_{i_1}, \dots, \sigma_{i_N}, p^{(i_1)}, \dots, p^{(i_N)}, \tilde{v}^{(i_N)})$  are kept fixed,

$$\text{Crit}((i_1 \dots i_N) \tilde{\psi}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}^{wk}) = \text{Crit}((i_1 \dots i_N) \tilde{\psi}^{wk}) \cap \left\{ \sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)} = \text{constant} \right\}.$$

Thus, the critical set of  $(i_1 \dots i_N) \tilde{\psi}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}^{wk}$  is equal to the fiber over  $(\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)})$  of the vector bundle

$$\left( (\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}), \mathfrak{g}_{\tilde{v}^{(i_N)}} \times \left( \bigoplus_{j=1}^N E^{(i_j)} \oplus F^{(i_N)} \right)^\perp \right) \mapsto (\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}),$$

and in particular a smooth submanifold. Lemma 2 then implies that the study of the transversal Hessian of  ${}^{(i_1 \dots i_N)}\tilde{\psi}^{wk}$  can be reduced to the study of the transversal Hessian of  ${}^{(i_1 \dots i_N)}\tilde{\psi}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}^{wk}$ . The crucial fact is now contained in the following

**Proposition 1.** *Assume that  $\sigma_{i_1} \dots \sigma_{i_N} = 0$ . Then*

$$\ker \text{Hess} {}^{(i_1 \dots i_N)}\tilde{\psi}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}^{wk}(0, \dots, 0, \beta^{(i_N)}, \xi) \simeq T_{(0, \dots, 0, \beta^{(i_N)}, \xi)} \text{Crit}({}^{(i_1 \dots i_N)}\tilde{\psi}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}^{wk})$$

for all  $(0, \dots, 0, \beta^{(i_N)}, \xi) \in \text{Crit}({}^{(i_1 \dots i_N)}\tilde{\psi}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}^{wk})$ , and arbitrary  $p^{(i_j)}, \tilde{v}^{(i_j)}$ .

*Proof.* Let us begin by computing

$$\begin{aligned} \partial_{\xi_r} {}^{(i_1 \dots i_N)}\tilde{\psi}^{wk} &= \left[ \sum_{j=1}^N \sum_{k=1}^{d^{(i_j)}} \alpha_k^{(i_j)} A_k^{(i_j)} p^{(i_j)} + \sum_{k=1}^{e^{(i_N)}} \beta_k^{(i_N)} B_k^{(i_N)} \tilde{v}^{(i_N)} \right]_r + \partial_{\xi_r} \sum_{j=1}^N O(|\tau_{i_j} A^{(i_j)}|) \\ &\quad + \partial_{\xi_r} O(|\tau_{i_N} B^{(i_N)} \tilde{v}^{(i_N)}|). \end{aligned}$$

If  $\sigma_{i_1} \dots \sigma_{i_j} = 0$ , which means that all the  $\tau_{i_j}$  are zero, the second derivatives read

$$\begin{aligned} \partial_{\xi_r} \partial_{\xi_s} {}^{(i_1 \dots i_N)}\tilde{\psi}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}^{wk} &= 0, \\ \partial_{\alpha_s^{(i_j)}} \partial_{\xi_r} {}^{(i_1 \dots i_N)}\tilde{\psi}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}^{wk} &= [A_s^{(i_j)} p^{(i_j)}]_r, \\ \partial_{\beta_s^{(i_N)}} \partial_{\xi_r} {}^{(i_1 \dots i_N)}\tilde{\psi}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}^{wk} &= [B_s^{(i_N)} \tilde{v}^{(i_N)}]_r. \end{aligned}$$

Next, one has

$$\partial_{\alpha_s^{(i_j)}} {}^{(i_1 \dots i_N)}\tilde{\psi}^{wk} = \left\langle A_s^{(i_j)} p^{(i_j)}, \xi \right\rangle + \partial_{\alpha_s^{(i_j)}} O(|\tau_{i_j} A^{(i_j)}|),$$

so that for  $\sigma_{i_1} \dots \sigma_{i_j} = 0$  all the second order derivatives involving  $\alpha^{(i_j)}$  must vanish, except the ones that were already computed. Finally, the computation of the  $\beta^{(i_N)}$ -derivatives yields

$$\partial_{\beta_r^{(i_N)}} \partial_{\beta_s^{(i_N)}} {}^{(i_1 \dots i_N)}\tilde{\psi}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}^{wk} = 0.$$

Collecting everything we see that the Hessian of the function  ${}^{(i_1 \dots i_N)}\tilde{\psi}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}^{wk}$  with respect to the coordinates  $\alpha^{(i_j)}, \beta^{(i_j)}, \xi$  is given on its critical set by the matrix

$$\begin{pmatrix} 0 & [A_s^{(i_1)} p^{(i_1)}]_r & \dots & [A_s^{(i_N)} p^{(i_N)}]_r & [B_s^{(i_N)} \tilde{v}^{(i_N)}]_r \\ [A_r^{(i_1)} p^{(i_1)}]_s & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ [A_r^{(i_N)} p^{(i_N)}]_s & 0 & \dots & 0 & 0 \\ [B_r^{(i_N)} \tilde{v}^{(i_N)}]_s & 0 & \dots & 0 & 0 \end{pmatrix},$$

where  $\sigma_{i_1} \dots \sigma_{i_j} = 0$ . Let us now compute the kernel of the linear transformation corresponding to this matrix. Clearly, the vector  $(\tilde{\xi}, \tilde{\alpha}^{(i_1)}, \dots, \tilde{\alpha}^{(i_N)}, \tilde{\beta}^{(i_N)})$  lies in the kernel if and only if

- $\sum \tilde{\alpha}_r^{(i_1)} A_r^{(i_1)} p^{(i_1)} + \dots + \sum \tilde{\alpha}_r^{(i_N)} A_r^{(i_N)} p^{(i_N)} + \sum \tilde{\beta}_r^{(i_N)} B_r^{(i_N)} \tilde{v}^{(i_N)} = 0$ ;
- $\left\langle Y^{(i_j)} p^{(i_j)}, \tilde{\xi} \right\rangle = 0$  for all  $Y^{(i_j)} \in \mathfrak{g}_{p^{(i_j)}}^\perp$ ,  $1 \leq j \leq N$ ;
- $\left\langle Z \tilde{v}^{(i_N)}, \tilde{\xi} \right\rangle = 0$  for all  $Z \in \mathfrak{g}_{\tilde{v}^{(i_N)}}^\perp$ .

Let  $V^{(i_1 \dots i_N)}$ ,  $E^{(i_j)}$  and  $F^{(i_N)}$  be the subspaces in  $\mathbb{R}^n$  defined in (6) and (12). Since  $\mathfrak{g}_{p^{(i_j)}}^\perp \subset \mathfrak{g}_{p^{(i_{j-1})}}$ , the condition (b) is equivalent to  $\tilde{\xi} \in V^{(i_1 \dots i_N)}$ . Next, we have

$$\sum \tilde{\alpha}_r^{(i_j)} A_r^{(i_1)} p^{(i_1)} + \dots + \sum \tilde{\alpha}_r^{(i_N)} A_r^{(i_N)} p^{(i_N)} + \sum \tilde{\beta}_r^{(i_N)} B_r^{(i_N)} \tilde{v}^{(i_N)} \in \bigoplus_{j=1}^N E^{(i_j)} \oplus F^{(i_N)},$$

so that for condition (a) to hold, it is necessary and sufficient that

$$\tilde{\alpha}^{(i_j)} = 0, \quad 1 \leq j \leq N, \quad \sum \tilde{\beta}_r^{(i_N)} B_r^{(i_N)} \tilde{v}^{(i_N)} = 0.$$

In addition, condition (c) is equivalent to

$$\tilde{\xi} \in N_{\tilde{v}^{(i_N)}}(G_{p^{(i_N)}} \cdot \tilde{v}^{(i_N)}).$$

On the other hand,

$$\begin{aligned} & T_{(0, \dots, 0, \beta^{(i_N)}, \xi)} \text{Crit} \left( \binom{i_1 \dots i_N}{\sigma_{i_j, p^{(i_j)}, \tilde{v}^{(i_N)}}} \tilde{\psi}^{wk} \right) \\ &= \left\{ (\tilde{\alpha}^{(i_1)}, \dots, \tilde{\alpha}^{(i_N)}, \tilde{\beta}^{(i_N)}, \tilde{\xi}) : \tilde{\alpha}^{(i_j)} = 0, \sum \tilde{\beta}_r^{(i_N)} B_r^{(i_N)} \in \mathfrak{g}_{\tilde{v}^{(i_N)}}, \tilde{\xi} \perp \bigoplus_{j=1}^N E^{(i_j)} \oplus F^{(i_N)} \right\}. \end{aligned}$$

But since for  $\sigma_{i_1} \dots \sigma_{i_N} = 0$

$$\left( \bigoplus_{j=1}^N E^{(i_j)} \oplus F^{(i_N)} \right)^\perp = N_{\tilde{v}^{(i_N)}}(G_{p^{(i_N)}} \cdot \tilde{v}^{(i_N)}) \cap V^{(i_1 \dots i_N)} = N_{\tilde{v}^{(i_N)}}(G_{p^{(i_N)}} \cdot \tilde{v}^{(i_N)}),$$

the proposition follows.  $\square$

Let now  $B$  be a symmetric bilinear form on a finite dimensional  $\mathbb{K}$ -vector space  $V$ , and  $M = (M_{ij})_{i,j}$  the corresponding Gramian matrix with respect to a basis  $\{v_1, \dots, v_n\}$  of  $V$  such that

$$B(u, w) = \sum_{i,j} u_i w_j M_{ij}, \quad u = \sum u_i v_i, \quad w = \sum w_i v_i.$$

We denote the linear operator given by  $M$  with the same letter, and write

$$V = \ker M \oplus W.$$

Consider the restriction  $B|_{W \times W}$  of  $B$  to  $W \times W$ , and assume that  $B|_{W \times W}(u, w) = 0$  for all  $u \in W$ , but  $w \neq 0$ . Since the Euclidean scalar product in  $V$  is non-degenerate, we necessarily must have  $Mw = 0$ , and consequently  $w \in \ker M \cap W = \{0\}$ , which is a contradiction. Therefore  $B|_{W \times W}$  defines a non-degenerate symmetric bilinear form. The previous proposition therefore implies that for  $\sigma_{i_1} \dots \sigma_{i_N} = 0$

$$\text{Hess} \binom{i_1 \dots i_N}{\sigma_{i_j, p^{(i_j)}, \tilde{v}^{(i_N)}}} \tilde{\psi}^{wk} (0, \dots, 0, \beta^{(i_N)}, \xi) \Big|_{N_{(0, \dots, 0, \beta^{(i_N)}, \xi)} \text{Crit} \left( \binom{i_1 \dots i_N}{\sigma_{i_j, p^{(i_j)}, \tilde{v}^{(i_N)}}} \tilde{\psi}^{wk} \right)}$$

defines a non-degenerate symmetric bilinear form for all points  $(0, \dots, 0, \beta^{(i_N)}, \xi)$  lying in the critical set of  $\binom{i_1 \dots i_N}{\sigma_{i_j, p^{(i_j)}, \tilde{v}^{(i_N)}}} \tilde{\psi}^{wk}$ . The second fundamental theorem now follows with Lemma 2.  $\square$

We are now in position to give an asymptotic description of the integral  $I(\mu)$ . But before, we shall say a few words about the desingularization process.

## 6. RESOLUTION OF SINGULARITIES AND THE STATIONARY PHASE THEOREM

Let  $M$  be a smooth variety,  $\mathcal{O}_M$  the structure sheaf of rings of  $M$ , and  $I \subset \mathcal{O}_M$  an ideal sheaf. The aim in the theory of resolution of singularities is to construct a birational morphism  $\pi : \tilde{M} \rightarrow M$  such that  $\tilde{M}$  is smooth, and the pulled back ideal sheaf  $\pi^*I$  is locally principal. This is called the *principalization* of  $I$ , and implies resolution of singularities. That is, for every quasi-projective variety  $X$ , there is a smooth variety  $\tilde{X}$ , and a birational and projective morphism  $\pi : \tilde{X} \rightarrow X$ . Vice versa, resolution of singularities implies principalization.

Consider next the derivative  $D(I)$  of  $I$ , which is the sheaf ideal that is generated by all derivatives of elements of  $I$ . Let further  $Z \subset M$  be a smooth subvariety, and  $\pi : B_Z M \rightarrow M$  the corresponding monoidal transformation with exceptional divisor  $F \subset B_Z M$ . Assume that  $(I, m)$  is a marked ideal sheaf with  $m \leq \text{ord}_Z I$ . The total transform  $\pi^*I$  vanishes along  $F$  with multiplicity  $\text{ord}_Z I$ , and by removing the ideal sheaf  $\mathcal{O}_{B_Z M}(-\text{ord}_Z I \cdot F)$  from  $\pi^*I$  we obtain the *birational, or weak transform*  $\pi_*^{-1}I$  of  $I$ . Take local coordinates  $(x_1, \dots, x_n)$  on  $M$  such that  $Z = (x_1 = \dots = x_r = 0)$ . As a consequence,

$$y_1 = \frac{x_1}{x_r}, \dots, y_{r-1} = \frac{x_{r-1}}{x_r}, y_r = x_r, \dots, y_n = x_n$$

define local coordinates on  $B_Z M$ , and for  $(f, m) \in (I, m)$  one has

$$\pi_*^{-1}(f(x_1, \dots, x_n), m) = (y_r^{-m} f(y_1 y_r, \dots, y_{r-1} y_r, y_r, \dots, y_n), m).$$

By computing the first derivatives of  $\pi_*^{-1}(f(x_1, \dots, x_n), m)$ , one then sees that for any composition  $\Pi : \tilde{M} \rightarrow M$  of blowing-ups of order greater or equal than  $m$ ,

$$\Pi_*^{-1}(D(I, m)) \subset D(\Pi_*^{-1}(I, m)),$$

see Kollár [10], Theorem 71.

Let us now come back to our situation, and consider on  $T^*\mathbb{R}^n \times \mathfrak{g}$  the ideal  $I_\psi = (\psi)$  generated by the phase function  $\psi = \mathbb{J}(x, \xi)(X) = \langle Xx, \xi \rangle$ , together with its vanishing set  $V_\psi$ . The derivative of  $I$  is given by  $D(I_\psi) = I_{\mathcal{C}}$ , where  $I_{\mathcal{C}}$  denotes the vanishing ideal of the critical set  $\mathcal{C} = \text{Crit}(\psi)$ , and by the implicit function theorem  $\text{Sing } V_\psi \subset V_\psi \cap \mathcal{C} = \mathcal{C}$ . Let  $((H_{i_1}), \dots, (H_{i_{N+1}}) = (H_L))$  be an arbitrary branch of isotropy types, and consider the corresponding sequence of monoidal transformations  $(\zeta_{i_1} \circ \zeta_{i_1 i_2} \circ \dots \circ \zeta_{i_1 \dots i_N}) \otimes \text{id}_\xi$ . Compose it with the sequence of monoidal transformations  $\delta_{i_1 \dots i_N}$ , and denote the resulting transformation by  $\zeta$ . We then have the diagram

$$\begin{array}{ccc} \zeta^*(I_{\mathcal{C}}) & \supset & \zeta^*(I_\psi) = \prod_{i=1}^N \tau_{i_j}(\sigma) \cdot \zeta_*^{-1}(I_\psi) \ni \tau_{i_1}(\sigma) \dots \tau_{i_N}(\sigma) \binom{i_1 \dots i_N}{\tilde{\psi}^{wk}} \\ \uparrow & & \uparrow \\ I_{\mathcal{C}} & \supset & I_\psi \ni \psi \end{array}$$

According to the previous considerations, we have the inclusion

$$\zeta_*^{-1}(I_{\mathcal{C}}) \subset D(\zeta_*^{-1}(I_\psi)).$$

Furthermore, the first fundamental theorem implies that  $D(\zeta_*^{-1}(I_\psi))$  is a resolved ideal. Nevertheless, it is easy to see that  $\zeta_*^{-1}(I_\psi)$  is not resolved, so that  $\prod_{i=1}^N \tau_{i_j}(\sigma) \cdot \zeta_*^{-1}(I_\psi)$  is only a partial principalization. Let us now consider the set

$$\begin{aligned} \tilde{\mathcal{C}}^{(i_1 \dots i_N)} &= \text{Cl} \left\{ (\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}, \alpha^{(i_j)}, \beta^{(i_N)}, \xi) : \sigma_{i_1} \dots \sigma_{i_N} \neq 0, \quad (x^{(i_1 \dots i_N)}, \xi, X^{(i_1 \dots i_N)}) \in \mathcal{C} \right\} \\ &= \text{Cl} \left\{ (\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}, 0, \beta^{(i_N)}, \xi) : \sigma_{i_1} \dots \sigma_{i_N} \neq 0, \quad B^{(i_N)} \in \mathfrak{g}_{(\tilde{v}^{(i_N)}, \xi)}, \quad \xi \perp (\mathfrak{g} \cdot x^{(i_1 \dots i_N)}) \right\} \\ &= \text{Cl} \left\{ (\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}, 0, \beta^{(i_N)}, \xi) : \sigma_{i_1} \dots \sigma_{i_N} \neq 0, \quad B^{(i_N)} \in \mathfrak{g}_{\tilde{v}^{(i_N)}}, \quad \xi \perp \bigoplus_{l=1}^N E^{(i_l)} \oplus F^{(i_N)} \right\}, \end{aligned}$$

where we made use of the decomposition

$$\mathfrak{g} \cdot x^{(i_1 \dots i_N)} = E^{(i_1)} \oplus \tau_{i_1} E^{(i_2)} \oplus \tau_{i_1} \sin \tau_{i_2} E^{(i_3)} \oplus \dots \oplus \tau_{i_1} \sin \tau_{i_2} \dots \sin \tau_{i_N} F^{(i_N)}$$

and took into account that  $G_{\tilde{v}^{(i_N)}}$  acts trivially on  $(\bigoplus_{l=1}^N E^{(i_l)} \oplus F^{(i_N)})^\perp$ . Equation (13) then implies that

$$\tilde{\mathcal{C}}^{(i_1 \dots i_N)} = \text{Crit}({}^{(i_1 \dots i_N)}\tilde{\psi}^{wk}).$$

Nevertheless, this does not result in a resolution  $\tilde{\mathcal{C}}$  of  $\mathcal{C}$ , but only in a partial resolution, since the induced global birational transform  $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$  is not surjective in general. This is because the centers of our monoidal transformations were only chosen in  $\mathbb{R}_x^n \times \mathfrak{g}$ , to keep the phase analysis of the weak transform of  $\psi$  as simple as possible. In turn, the  $\xi$ -singularities of  $\mathcal{C}$  were not completely resolved.

As we shall see in the next section, the principalization of the ideal  $I_\psi$

$$\zeta^*(I_\psi) = \tau_{i_1} \dots \tau_{i_N} \zeta_*^{-1}(I_\psi),$$

and the fact that the weak transform  $({}^{i_1 \dots i_N}\tilde{\psi}^{wk})$  has a clean critical set, are essential for an application of the stationary phase principle in the context of singular equivariant asymptotics. By Hironaka's theorem on resolution of singularities, a resolution  $\zeta$  of the vanishing set of  $\psi$  always exists, which is equivalent to the principalization of the ideal  $I_\psi$ . But in general, such a resolution would not be explicit enough<sup>4</sup> to allow an application of the stationary phase theorem. This is the reason why we were forced to construct an explicit, though partial, resolution  $\zeta$  of  $\mathcal{C}$  in  $T^*\mathbb{R}^n \times \mathfrak{g}$ , using as centers isotropy algebra bundles over sets of maximal singular orbits. Partial desingularizations of the zero level set  $\Omega$  of the moment map and the symplectic quotient  $\Omega/G$  have been obtained e.g. by Meinrenken-Sjamaar [11] for compact symplectic manifolds with a Hamiltonian compact Lie group action by performing blowing-ups along minimal symplectic suborbifolds containing the strata of maximal depth in  $\Omega$ .

## 7. ASYMPTOTICS FOR THE INTEGRALS $I_{i_1 \dots i_N}(\mu)$

In this section, we will give an asymptotic description of the integrals  $I_{i_1 \dots i_N}(\mu)$  defined in (3). Since the considered integrals are absolutely convergent integral, we can interchange the order of integration by Fubini, and write

$$I_{i_1 \dots i_N}(\mu) = \int_{(-T, T)^N} J_{\tau_{i_1}, \dots, \tau_{i_N}} \left( \frac{\mu}{\tau_{i_1} \dots \tau_{i_N}} \right) \prod_{j=1}^N |\tau_{i_j}|^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - 1} d\tau_{i_N} \dots d\tau_{i_1},$$

where we set

$$J_{\tau_{i_1}, \dots, \tau_{i_N}}(\nu) = \int e^{i({}^{i_1 \dots i_N}\tilde{\psi}^{wk, pre})/\nu} a_{i_1 \dots i_N} \Phi_{i_1 \dots i_N} d\xi dA^{(i_1)} \dots dA^{(i_N)} dB^{(i_N)} d\tilde{v}^{(i_N)} dp^{(i_N)} \dots dp^{(i_1)},$$

and introduced the new parameter

$$\nu = \frac{\mu}{\tau_{i_1} \dots \tau_{i_N}}.$$

Now, for an arbitrary  $0 < \varepsilon < T$  to be chosen later we define

$$I_{i_1 \dots i_N}^1(\mu) = \int_{((-T, T) \setminus (-\varepsilon, \varepsilon))^N} J_{\tau_{i_1}, \dots, \tau_{i_N}} \left( \frac{\mu}{\tau_{i_1} \dots \tau_{i_N}} \right) \prod_{j=1}^N |\tau_{i_j}|^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - 1} d\tau_{i_N} \dots d\tau_{i_1},$$

$$I_{i_1 \dots i_N}^2(\mu) = \int_{(-\varepsilon, \varepsilon)^N} J_{\tau_{i_1}, \dots, \tau_{i_N}} \left( \frac{\mu}{\tau_{i_1} \dots \tau_{i_N}} \right) \prod_{j=1}^N |\tau_{i_j}|^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - 1} d\tau_{i_N} \dots d\tau_{i_1}.$$

<sup>4</sup>In particular, the so-called numerical data of  $\zeta$  are not known a priori, which in our case are given in terms of the dimensions  $c^{(i_j)}$  and  $d^{(i_j)}$ .

**Lemma 3.** *One has  $c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - 1 \geq \kappa$  for arbitrary  $j = 1, \dots, N$ .*

*Proof.* We first note that

$$c^{(i_j)} = \dim(\nu_{i_1 \dots i_j})_{p^{(i_j)}} \geq \dim G_{p^{(i_j)}} \cdot x^{(i_{j+1} \dots i_N)} + 1.$$

Indeed,  $(\nu_{i_1 \dots i_j})_{p^{(i_j)}}$  is an orthogonal  $G_{p^{(i_j)}}$ -space, so that the dimension of the  $G_{p^{(i_j)}}$ -orbit of  $x^{(i_{j+1} \dots i_N)} \in (S_{i_1 \dots i_j}^+)_{p^{(i_j)}}$  can be at most  $c^{(i_j)} - 1$ . Now, under the assumption  $\sigma_{i_1} \dots \sigma_{i_N} \neq 0$  one computes

$$\begin{aligned} \dim G_{p^{(i_j)}} \cdot x^{(i_{j+1} \dots i_N)} &= \dim \mathfrak{g}_{p^{(i_j)}} \cdot x^{(i_{j+1} \dots i_N)} = \dim[\mathfrak{g}_{p^{(i_N)}} \oplus \mathfrak{g}_{p^{(i_N)}}^\perp \oplus \dots \oplus \mathfrak{g}_{p^{(i_{j+1})}}^\perp] \cdot x^{(i_{j+1} \dots i_N)} \\ &= \dim[\mathfrak{g}_{p^{(i_{j+1})}}^\perp \cdot x^{(i_{j+1} \dots i_N)} + \sin \tau_{i_{j+1}} \mathfrak{g}_{p^{(i_{j+2})}}^\perp \cdot x^{(i_{j+2} \dots i_N)} + \dots \\ &\quad + \sin \tau_{i_{j+1}} \dots \sin \tau_{i_{N-1}} \mathfrak{g}_{p^{(i_N)}}^\perp \cdot x^{(i_N)} + \sin \tau_{i_{j+1}} \dots \sin \tau_{i_N} \mathfrak{g}_{p^{(i_N)}} \tilde{v}^{(i_N)}] \\ &= \dim[E^{(i_{j+1})} \oplus \sin \tau_{i_{j+1}} E^{(i_{j+2})} \oplus \dots \oplus \sin \tau_{i_{j+1}} \dots \sin \tau_{i_{N-1}} E^{(i_N)} \oplus \sin \tau_{i_{j+1}} \dots \sin \tau_{i_N} F^{(i_N)}] \\ &= \sum_{l=j+1}^N \dim E^{(i_l)} + \dim F^{(i_N)}, \end{aligned}$$

which implies

$$c^{(i_j)} \geq \sum_{l=j+1}^N \dim E^{(i_l)} + \dim F^{(i_N)} + 1.$$

Here we used the same arguments as in the proof of Equation (14). On the other hand, one has

$$d^{(i_j)} = \dim \mathfrak{g}_{p^{(i_j)}}^\perp = \dim \mathfrak{g}_{p^{(i_j)}}^\perp \cdot p^{(i_j)} = \dim \mathfrak{g}_{p^{(i_j)}}^\perp \cdot x^{(i_j \dots i_N)} = \dim E^{(i_j)},$$

since  $x^{(i_j \dots i_N)}$  lies in a slice around  $G_{p^{(i_{j-1})}} \cdot p^{(i_j)}$ . The assertion now follows with (14).  $\square$

As a consequence of the lemma, we obtain for  $I_{i_1 \dots i_N}^2(\mu)$  the estimate

$$\begin{aligned} (17) \quad I_{i_1 \dots i_N}^2(\mu) &\leq C \int_{(-\varepsilon, \varepsilon)^N} \prod_{j=1}^N |\tau_{i_j}|^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - 1} d\tau_{i_N} \dots d\tau_{i_1} \\ &\leq C \int_{(-\varepsilon, \varepsilon)^N} \prod_{j=1}^N |\tau_{i_j}|^\kappa d\tau_{i_N} \dots d\tau_{i_1} = \frac{2C}{\kappa + 1} \varepsilon^{N(\kappa+1)} \end{aligned}$$

for some  $C > 0$ . Let us now turn to the integral  $I_{i_1 \dots i_N}^1(\mu)$ . After performing the change of variables  $\delta_{i_1 \dots i_N}$  one obtains

$$I_{i_1 \dots i_N}^1(\mu) = \int_{\varepsilon < |\tau_{i_j}(\sigma)| < T} J_{\sigma_{i_1}, \dots, \sigma_{i_N}} \left( \frac{\mu}{\tau_{i_1}(\sigma) \dots \tau_{i_N}(\sigma)} \right) \prod_{j=1}^N |\tau_{i_j}(\sigma)|^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - 1} |\det D\delta_{i_1 \dots i_N}(\sigma)| d\sigma,$$

where

$$J_{\sigma_{i_1}, \dots, \sigma_{i_N}}(\nu) = \int e^{i(i_1 \dots i_N) \tilde{\psi}_\sigma^{wk} / \nu} a_{i_1 \dots i_N} \Phi_{i_1 \dots i_N} d\xi dA^{(i_1)} \dots dA^{(i_N)} dB^{(i_N)} d\tilde{v}^{(i_N)} dp^{(i_N)} \dots dp^{(i_1)}.$$

Here we denoted by  $(i_1 \dots i_N) \tilde{\psi}_\sigma^{wk}$  the weak transform of the phase function  $\psi$  as a function of the variables  $p^{(i_j)}, \tilde{v}^{(i_N)}, \alpha^{(i_j)}, \beta^{(i_N)}$  alone, while the variables  $\sigma = (\sigma_{i_1}, \dots, \sigma_{i_N})$  are regarded as parameters. The idea is now to make use of the principle of the stationary phase to give an asymptotic expansion of  $J_{\sigma_{i_1}, \dots, \sigma_{i_N}}(\nu)$ .

**Theorem 3** (Generalized stationary phase theorem for manifolds). *Let  $M$  be a  $n$ -dimensional Riemannian manifold,  $\psi \in C^\infty(M)$  be a real valued phase function,  $\mu > 0$ , and set*

$$I(\mu) = \int_M e^{i\psi(m)/\mu} a(m) dm,$$

where  $a(m)dm$  denotes a compactly supported  $C^\infty$ -density on  $M$ . Let

$$\mathcal{C} = \{m \in M : \psi_* : TM_m \rightarrow T\mathbb{R}_{\psi(m)} \text{ is zero}\}$$

be the critical set of the phase function  $\psi$ , and assume that

- (1)  $\mathcal{C}$  is a smooth submanifold of  $M$  of dimension  $p$  in a neighborhood of the support of  $a$ ;
- (2) for all  $m \in \mathcal{C}$ , the restriction  $\psi''(m)|_{N_m\mathcal{C}}$  of the Hessian of  $\psi$  at the point  $m$  to the normal space  $N_m\mathcal{C}$  is a non-degenerate quadratic form.

Then, for all  $N \in \mathbb{N}$ , there exists a constant  $C_{N,\psi} > 0$  such that

$$|I(\mu) - e^{i\psi_0/\mu} (2\pi\mu)^{\frac{n-p}{2}} \sum_{j=0}^{N-1} \mu^j Q_j(\psi; a)| \leq C_{N,\psi} \mu^N \text{vol}(\text{supp } a \cap \mathcal{C}) \sup_{l \leq 2N} \|D^l a\|_{\infty, M},$$

where  $D^l$  is a differential operator on  $M$  of order  $l$ , and  $\psi_0$  is the constant value of  $\psi$  on  $\mathcal{C}$ . Furthermore, for each  $j$  there exists a constant  $\tilde{C}_{j,\psi} > 0$  such that

$$|Q_j(\psi; a)| \leq \tilde{C}_{j,\psi} \text{vol}(\text{supp } a \cap \mathcal{C}) \sup_{l \leq 2j} \|D^l a\|_{\infty, \mathcal{C}},$$

and, in particular,

$$Q_0(\psi; a) = \int_{\mathcal{C}} \frac{a(m)}{|\det \psi''(m)|_{N_m\mathcal{C}}|^{1/2}} d\sigma_{\mathcal{C}}(m) e^{i\pi\sigma_{\psi''}},$$

where  $\sigma_{\psi''}$  is the constant value of the signature of  $\psi''(m)|_{N_m\mathcal{C}}$  for  $m$  in  $\mathcal{C}$ .

*Proof.* See for instance Hörmander, [8], Theorem 7.7.5, together with Combescure-Ralston-Robert [5], Theorem 3.3.  $\square$

**Remark 1.** An examination of the proof of the foregoing theorem shows that the constants  $C_{N,\psi}$  are essentially bounded from above by

$$\sup_{m \in \mathcal{C} \cap \text{supp } a} \left\| \left( \psi''(m)|_{N_m\mathcal{C}} \right)^{-1} \right\|.$$

Indeed, let  $\alpha : (x, y) \rightarrow m \in \mathcal{O} \subset M$  be local normal coordinates such that  $\alpha(x, y) \in \mathcal{C}$  if, and only if,  $y = 0$ . By (15), the transversal Hessian  $\text{Hess } \psi(m)|_{N_m\mathcal{C}}$  is given in these coordinates by the matrix

$$\left( \partial_{y_k} \partial_{y_l} (\psi \circ \alpha)(x, 0) \right)_{k,l}$$

where  $m = \alpha(x, 0)$ . If now the transversal Hessian of  $\psi$  is non-degenerate at the point  $m = \alpha(x, 0)$ , then  $y = 0$  is a non-degenerate critical point of the function  $y \mapsto (\psi \circ \alpha)(x, y)$ , and therefore an isolated critical point by the lemma of Morse. As a consequence,

$$(18) \quad \frac{|y|}{|\partial_y (\psi \circ \alpha)(x, y)|} \leq 2 \left\| \left( \partial_{y_k} \partial_{y_l} (\psi \circ \alpha)(x, 0) \right)_{k,l}^{-1} \right\|$$

for  $y$  close to zero. The assertion now follows by applying Hörmander [8], Theorem 7.7.5, to the integral

$$\int_{\alpha^{-1}(\mathcal{O})} e^{i(\psi \circ \alpha)(x, y)/\mu} (a \circ \alpha)(x, y) dy dx$$

in the variable  $y$ , and with  $x$  as a parameter, since in our situation the constant  $C$  occurring in Hörmander [8], Equation (7.7.12), is precisely bounded by (18), if we assume as we may that  $a$  is supported near  $\mathcal{C}$ . A similar observation holds with respect to the constants  $\tilde{C}_{j,\psi}$ .

Now, as a consequence of the fundamental theorems, and Lemma 2, together with the observations preceding Proposition 1,

- the critical set  $\text{Crit}^{((i_1 \dots i_N) \tilde{\psi}_\sigma^{wk})}$  is a  $C^\infty$ -submanifold of codimension  $2\kappa$  for arbitrary  $\sigma$ ;
- the transversal Hessian

$$\text{Hess}^{(i_1 \dots i_N) \tilde{\psi}_\sigma^{wk}}(p^{(i_j)}, \tilde{v}^{(i_N)}, \alpha^{(i_j)}, \beta^{(i_N)}) \Big|_{N_{(p^{(i_j)}, \tilde{v}^{(i_N)}, \alpha^{(i_j)}, \beta^{(i_N)})} \text{Crit}^{((i_1 \dots i_N) \tilde{\psi}_\sigma^{wk})}}$$

defines a non-degenerate symmetric bilinear form for arbitrary  $\sigma$  at every point of the critical set of  $(i_1 \dots i_N) \tilde{\psi}_\sigma^{wk}$ .

Thus, the necessary conditions for applying the principle of the stationary phase to the integral  $J_{\sigma_{i_1}, \dots, \sigma_{i_N}}(\nu)$  are fulfilled, and we arrive at the following

**Theorem 4.** *Let  $\sigma = (\sigma_{i_1}, \dots, \sigma_{i_N})$  be a fixed set of parameters. Then, for every  $\tilde{N} \in \mathbb{N}$  there exists a constant  $C_{\tilde{N}, (i_1 \dots i_N) \tilde{\psi}_\sigma^{wk}} > 0$  such that*

$$|J_{\sigma_{i_1}, \dots, \sigma_{i_N}}(\nu) - (2\pi|\nu|)^\kappa \sum_{j=0}^{\tilde{N}-1} |\nu|^j Q_j((i_1 \dots i_N) \tilde{\psi}_\sigma^{wk}; a_{i_1 \dots i_N} \Phi_{i_1 \dots i_N})| \leq C_{\tilde{N}, (i_1 \dots i_N) \tilde{\psi}_\sigma^{wk}} |\nu|^{\tilde{N}},$$

with estimates for the coefficients  $Q_j$ , and an explicit expression for  $Q_0$ .

□

Before going on, let us remark that for the computation of the integrals  $I_{i_1 \dots i_N}^1(\mu)$  it is only necessary to have an asymptotic expansion for the integrals  $J_{\sigma_{i_1}, \dots, \sigma_{i_N}}(\nu)$  in the case that  $\sigma_{i_1} \dots \sigma_{i_N} \neq 0$ , which can also be obtained without the fundamental theorems using only the factorization of the phase function  $\psi$  given by the resolution process. Nevertheless, the main consequence to be drawn from the fundamental theorems is that the constants  $C_{\tilde{N}, (i_1 \dots i_N) \tilde{\psi}_\sigma^{wk}}$  and the coefficients  $Q_j$  in Theorem 4 have uniform bounds in  $\sigma$ . Indeed, by Remark 1 we have

$$C_{\tilde{N}, (i_1 \dots i_N) \tilde{\psi}_\sigma^{wk}} \leq C'_{\tilde{N}} \sup_{p^{(i_j)}, \tilde{v}^{(i_N)}, \alpha^{(i_j)}, \beta^{(i_N)}} \left\| \left( \text{Hess}^{(i_1 \dots i_N) \tilde{\psi}_\sigma^{wk}} \Big|_{N \text{Crit}^{((i_1 \dots i_N) \tilde{\psi}_\sigma^{wk})}} \right)^{-1} \right\|.$$

But since by Lemma 2 the transversal Hessian

$$\text{Hess}^{(i_1 \dots i_N) \tilde{\psi}_\sigma^{wk}} \Big|_{N_{(p^{(i_j)}, \tilde{v}^{(i_N)}, \alpha^{(i_j)}, \beta^{(i_N)})} \text{Crit}^{((i_1 \dots i_N) \tilde{\psi}_\sigma^{wk})}}$$

is given by

$$\text{Hess}^{(i_1 \dots i_N) \tilde{\psi}_\sigma^{wk}} \Big|_{N_{(\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}, \alpha^{(i_j)}, \beta^{(i_N)})} \text{Crit}^{((i_1 \dots i_N) \tilde{\psi}_\sigma^{wk})}},$$

we finally obtain the estimate

$$C_{\tilde{N}, (i_1 \dots i_N) \tilde{\psi}_\sigma^{wk}} \leq C'_{\tilde{N}} \sup_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}, \alpha^{(i_j)}, \beta^{(i_N)}} \left\| \left( \text{Hess}^{(i_1 \dots i_N) \tilde{\psi}_\sigma^{wk}} \Big|_{N \text{Crit}^{((i_1 \dots i_N) \tilde{\psi}_\sigma^{wk})}} \right)^{-1} \right\| \leq C_{\tilde{N}, (i_1 \dots i_N)}$$

by a constant independent of  $\sigma$ . Similarly, one can show the existence of bounds of the form

$$|Q_j^{((i_1 \dots i_N) \tilde{\psi}_\sigma^{wk}; a_{i_1 \dots i_N} \Phi_{i_1 \dots i_N})}| \leq \tilde{C}_{j, (i_1 \dots i_N)},$$



with constants  $\tilde{C}_{j,(i_1\dots i_N)}$  independent of  $\sigma$ . As a consequence of Theorem 4, we obtain for arbitrary  $\tilde{N} \in \mathbb{N}$

$$\begin{aligned} & |J_{\sigma_{i_1}, \dots, \sigma_{i_N}}(\nu) - (2\pi|\nu|)^\kappa Q_0((i_1\dots i_N)\tilde{\psi}_\sigma^{wk}; a_{i_1\dots i_N}\Phi_{i_1\dots i_N})| \\ & \leq \left| J_{\sigma_{i_1}, \dots, \sigma_{i_N}}(\nu) - (2\pi|\nu|)^\kappa \sum_{l=0}^{\tilde{N}-1} |\nu|^l Q_l((i_1\dots i_N)\tilde{\psi}_\sigma^{wk}; a_{i_1\dots i_N}\Phi_{i_1\dots i_N}) \right| \\ & + (2\pi|\nu|)^\kappa \sum_{l=1}^{\tilde{N}-1} |\nu|^l |Q_l((i_1\dots i_N)\tilde{\psi}_\sigma^{wk}; a_{i_1\dots i_N}\Phi_{i_1\dots i_N})| \leq c_1 |\nu|^{\tilde{N}} + c_2 |\nu|^\kappa \sum_{l=1}^{\tilde{N}-1} |\nu|^l \end{aligned}$$

with constants  $c_i > 0$  independent of both  $\sigma$  and  $\nu$ . From this we deduce

$$\begin{aligned} & \left| I_{i_1\dots i_N}^1(\mu) - (2\pi\mu)^\kappa \int_{\varepsilon < |\tau_{i_j}(\sigma)| < T} Q_0 \prod_{j=1}^N |\tau_{i_j}(\sigma)|^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - 1 - \kappa} |\det D\delta_{i_1\dots i_N}(\sigma)| d\sigma \right| \\ & \leq c_3 \mu^{\tilde{N}} \int_{\varepsilon < |\tau_{i_j}(\sigma)| < T} \prod_{j=1}^N |\tau_{i_j}(\sigma)|^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - 1 - \tilde{N}} |\det D\delta_{i_1\dots i_N}(\sigma)| d\sigma \\ & + c_4 \mu^\kappa \sum_{l=1}^{\tilde{N}-1} \mu^l \int_{\varepsilon < |\tau_{i_j}(\sigma)| < T} \prod_{j=1}^N |\tau_{i_j}(\sigma)|^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - 1 - \kappa - l} |\det D\delta_{i_1\dots i_N}(\sigma)| d\sigma \\ & \leq c_5 \mu^{\tilde{N}} \max \left\{ 1, \prod_{j=1}^N \varepsilon^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - \tilde{N}} \right\} + c_6 \sum_{l=1}^{\tilde{N}-1} \mu^{\kappa+l} \max \left\{ 1, \prod_{j=1}^N \varepsilon^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - \kappa - l} \right\}. \end{aligned}$$

We now set

$$\varepsilon = \mu^{1/N}.$$

Taking into account Lemma 3, one infers that the right hand side of the last inequality can be estimated by

$$c_5 \max \left\{ \mu^{\tilde{N}}, \mu^{\kappa+1} \right\} + c_6 \sum_{l=1}^{\tilde{N}-1} \max \left\{ \mu^{\kappa+l}, \mu^{\kappa+1} \right\},$$

so that for sufficiently large  $\tilde{N} \in \mathbb{N}$  we finally obtain an asymptotic expansion for  $I_{i_1\dots i_N}(\mu)$  by taking into account (17), and the fact that

$$(2\pi\mu)^\kappa \int_{0 < |\tau_{i_j}| < \mu^{1/N}} Q_0 \prod_{j=1}^N |\tau_{i_j}|^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - 1} d\tau_{i_N} \dots d\tau_{i_1} = O(\mu^{\kappa+1}).$$

**Theorem 5.** *Let the assumptions of the first fundamental theorem be fulfilled. Then*

$$I_{i_1\dots i_N}(\mu) = (2\pi\mu)^\kappa L_{i_1\dots i_N} + O(\mu^{\kappa+1}),$$

where the leading coefficient  $L_{i_1\dots i_N}$  is given by

$$(19) \quad L_{i_1\dots i_N} = \int_{\text{Crit}((i_1\dots i_N)\tilde{\psi}^{wk})} \frac{a_{i_1\dots i_N} \Phi_{i_1\dots i_N} d\text{Crit}((i_1\dots i_N)\tilde{\psi}^{wk})}{|\text{Hess}((i_1\dots i_N)\tilde{\psi}^{wk})_{N\text{Crit}((i_1\dots i_N)\tilde{\psi}^{wk})}|^{1/2}}.$$

□

## 8. STATEMENT OF THE MAIN RESULT

Let us now return to our departing point, that is, the asymptotic behavior of the integral  $I(\mu)$  introduced in (1). For this, we still have to examine the contributions to  $I(\mu)$  coming from integrals of the form

$$(20) \quad \begin{aligned} \tilde{I}_{i_1 \dots i_\Theta}(\mu) &= \int_{M_{i_1}(H_{i_1}) \times (-T, T)} \left[ \int_{(S_{i_1})_{p(i_1), i_2}(H_{i_2}) \times (-T, T)} \dots \left[ \int_{(S_{i_1 \dots i_{\Theta-1}})_{p(i_{\Theta-1}), i_\Theta}(H_{i_\Theta}) \times (-T, T)} \right. \right. \\ &\left. \left[ \int_{(S_{i_1 \dots i_\Theta}^+)_{p(i_\Theta)} \times \mathfrak{g}_{p(i_\Theta)} \times \mathfrak{g}_{p(i_\Theta)}^\perp \times \dots \times \mathfrak{g}_{p(i_1)}^\perp \times \mathbb{R}^n} e^{i \frac{\tau_1 \dots \tau_\Theta}{\mu} (i_1 \dots i_\Theta) \tilde{\psi}^{wk}} a_{i_1 \dots i_\Theta} \tilde{\Phi}_{i_1 \dots i_\Theta} \right. \right. \\ &\left. \left. d\xi dA^{(i_1)} \dots dA^{(i_\Theta)} dB^{(i_\Theta)} d\tilde{v}^{(i_\Theta)} \right] d\tau_{i_\Theta} dp^{(i_\Theta)} \dots \right] d\tau_{i_2} dp^{(i_2)} \left. \right] d\tau_{i_1} dp^{(i_1)}. \end{aligned}$$

where

$$a_{i_1 \dots i_\Theta} = [a \chi_{i_1} \circ (\text{id}_\xi \otimes \zeta_{i_1} \circ \zeta_{i_1 i_2} \circ \dots \circ \zeta_{i_1 \dots i_\Theta})] [\chi_{i_1 i_2} \circ \zeta_{i_1 i_2} \circ \dots \circ \zeta_{i_1 \dots i_\Theta}] \dots [\chi_{i_1 \dots i_\Theta} \circ \zeta_{i_1 \dots i_\Theta}]$$

is supposed to have compact support in one of the  $\alpha^{(i_\Theta)}$ -charts, and

$$\tilde{\Phi}_{i_1 \dots i_\Theta} = \prod_{j=1}^{\Theta} |\tau_{i_j}|^{c^{(i_j)} + \sum_r^j d^{(i_r)} - 1} \Phi_{i_1 \dots i_\Theta},$$

$\Phi_{i_1 \dots i_\Theta}$  being a smooth function which does not depend on the variables  $\tau_{i_j}$ . Now, a computation of the  $\xi$ -derivatives of  $(i_1 \dots i_\Theta) \tilde{\psi}^{wk}$  in any of the  $\alpha^{(i_\Theta)}$ -charts shows that  $(i_1 \dots i_\Theta) \tilde{\psi}^{wk}$  has no critical points there. By the non-stationary phase theorem, see Hörmander [8], Theorem 7.7.1, one then computes for arbitrary  $\tilde{N} \in \mathbb{N}$

$$|\tilde{I}_{i_1 \dots i_\Theta}(\mu)| \leq c_7 \mu^{\tilde{N}} \int_{\varepsilon < |\tau_{i_j}| < T} \prod_{j=1}^{\Theta} |\tau_{i_j}|^{c^{(i_j)} + \sum_r^j d^{(i_r)} - 1 - \tilde{N}} d\tau + c_8 \varepsilon^{\Theta(\kappa+1)} \leq c_9 \max \left\{ \mu^{\tilde{N}}, \mu^{\kappa+1} \right\},$$

where we took  $\varepsilon = \mu^{1/\Theta}$ . Choosing  $\tilde{N}$  large enough, we conclude that

$$|\tilde{I}_{i_1 \dots i_\Theta}(\mu)| = O(\mu^{\kappa+1}).$$

As a consequence of this we see that, up to terms of order  $O(\mu^{\kappa+1})$ ,  $I(\mu)$  can be written as a sum

$$\begin{aligned} I(\mu) &= \sum_{k < L} I_k(\mu) + I_L(\mu) = \sum_{k < l < L} I_{kl}(\mu) + \sum_{k < L} I_{kL}(\mu) + I_L(\mu) \\ &= \sum_N \sum_{i_1 < \dots < i_N < i_{N+1} = L} I_{i_1 \dots i_N}(\mu) + \sum_M \sum_{i_1 < \dots < i_M < i_{M+1} \neq L} I_{i_1 \dots i_M L}(\mu), \end{aligned}$$

where the first term in the last line is a sum to be taken over all the indices  $i_1, \dots, i_N$  corresponding to all possible isotropy branches of the form  $(H_{i_1}, \dots, (H_{i_N}), (H_{i_{N+1}}) = (H_L))$  of varying length  $N$ , while the second term is a sum over all indices  $i_1, \dots, i_M$  corresponding to branches  $(H_{i_1}, \dots, (H_{i_M}), (H_{i_{M+1}}) \neq (H_L))$  of arbitrary length  $M$ . The asymptotic behavior of the integrals  $I_{i_1 \dots i_N}(\mu)$  has been determined in the previous section, and it is not difficult to see that the integrals  $I_{i_1 \dots i_M L}$  have analogous asymptotic descriptions. We are now ready to state and prove the main result of this paper.

**Theorem 6.** *Let  $G$  be a compact, connected Lie group  $G$  with Lie algebra  $\mathfrak{g}$ , acting orthogonally on Euclidean space  $\mathbb{R}^n$ , and define*

$$I(\mu) = \int_{T^* \mathbb{R}^n} \int_{\mathfrak{g}} e^{i\psi(x, \xi, X)/\mu} a(x, \xi, X) dX d\xi dx, \quad \mu > 0,$$

where the phase function

$$\psi(x, \xi, X) = \mathbb{J}(x, \xi)(X) = \langle Xx, \xi \rangle$$

is given by the moment map  $\mathbb{J} : T^*\mathbb{R}^n \rightarrow \mathfrak{g}^*$  of the underlying Hamiltonian action, and  $dx d\xi$ ,  $dX$  are Lebesgue measures in  $T^*\mathbb{R}^n$ , and  $\mathfrak{g}$ , respectively, and  $a \in C_c^\infty(T^*\mathbb{R}^n \times \mathfrak{g})$ . Then

$$I(\mu) = (2\pi\mu)^\kappa L_0 + O(\mu^{\kappa+1}), \quad \mu \rightarrow 0^+.$$

Here  $\kappa$  is the dimension of an orbit of principal type in  $\mathbb{R}^n$ , and the leading coefficient is given by

$$(21) \quad L_0 = \int_{\text{Reg } \mathcal{C}} \frac{a(x, \xi, X)}{|\text{Hess } \psi(x, \xi, X)|_{N_{(x, \xi, X)} \text{Reg } \mathcal{C}}|^{1/2}} d(\text{Reg } \mathcal{C})(x, \xi, X),$$

where  $\text{Reg } \mathcal{C}$  denotes the regular part of the critical set  $\mathcal{C} = \text{Crit}(\psi)$  of  $\psi$ . In particular, the integral over  $\text{Reg } \mathcal{C}$  exists.

**Remark 2.** Note that Equation (21) in particular means that the obtained asymptotic expansion for  $I(\mu)$  is independent of the explicit partial resolution we used.

*Proof.* By our previous considerations, one has

$$I(\mu) = (2\pi\mu)^\kappa L_0 + O(\mu^{\kappa+1}), \quad \mu \rightarrow 0^+,$$

where  $L_0$  is given as a sum of integrals of the form (19). It therefore remains to show the equality (21). For this let us introduce first certain cut-off functions for the singular part  $\text{Sing } \Omega$  of  $\Omega$ . Thus, let  $K$  be a compact subset in  $\mathbb{R}^{2n}$ ,  $\varepsilon > 0$ , and denote by  $v_\varepsilon$  the characteristic function of the set

$$(\text{Sing } \Omega \cap K)_{2\varepsilon} = \{z \in \mathbb{R}^{2n} : |z - z'| < 2\varepsilon \text{ for some } z' \in \text{Sing } \Omega \cap K\}.$$

Consider further the unit ball  $B_1$  in  $\mathbb{R}^{2n}$ , together with a function  $\iota \in C_c^\infty(B_1)$  with  $\int \iota dz = 1$ , and set  $\iota_\varepsilon(z) = \varepsilon^{-2n} \iota(z/\varepsilon)$ . Clearly  $\int \iota_\varepsilon dz = 1$ ,  $\text{supp } \iota_\varepsilon \subset B_\varepsilon$ , and we define

$$(22) \quad u_\varepsilon = v_\varepsilon * \iota_\varepsilon.$$

One can then show that  $u_\varepsilon \in C_c^\infty((\text{Sing } \Omega \cap K)_{3\varepsilon})$ , and  $u_\varepsilon = 1$  on  $(\text{Sing } \Omega \cap K)_\varepsilon$ , together with

$$|\partial_z^\alpha u_\varepsilon| \leq C_\alpha \varepsilon^{-|\alpha|},$$

where  $C_\alpha$  is a constant which depends only on  $\alpha$  and  $n$ , see Hörmander [8], Theorem 1.4.1.

Next, we shall prove

**Lemma 4.** *Let  $a \in C_c^\infty(\mathbb{R}^{2n} \times \mathfrak{g})$ , and  $K$  be such that  $\text{supp } a \subset K$ . Then the limit*

$$(23) \quad \lim_{\varepsilon \rightarrow 0} \int_{\text{Reg } \mathcal{C}} \frac{[a(1 - u_\varepsilon)](x, \xi, X)}{|\det \psi''(x, \xi, X)|_{N_{(x, \xi, X)} \text{Reg } \mathcal{C}}|^{1/2}} d(\text{Reg } \mathcal{C})(x, \xi, X)$$

*exists and is equal to  $L_0$ , where  $d(\text{Reg } \mathcal{C})$  is the induced Riemannian measure on  $\text{Reg } \mathcal{C}$ .*

*Proof.* With  $u_\varepsilon$  as in Equation (22), let us define

$$I_\varepsilon(\mu) = \int_{T^*\mathbb{R}^n} \int_{\mathfrak{g}} e^{\frac{i}{\mu} \psi(x, \xi, X)} [a(1 - u_\varepsilon)](x, \xi, X) dX d\xi dx.$$

Since  $(x, \xi, X) \in \text{Sing } \mathcal{C}$  implies  $(x, \xi) \in \text{Sing } \Omega$ , a direct application of the generalized theorem of the stationary phase for fixed  $\varepsilon > 0$  gives

$$(24) \quad |I_\varepsilon(\mu) - (2\pi\mu)^\kappa L_0(\varepsilon)| \leq C_\varepsilon \mu^{\kappa+1},$$

where  $C_\varepsilon > 0$  is a constant depending only on  $\varepsilon$ , and

$$L_0(\varepsilon) = \int_{\text{Reg } \mathcal{C}} \frac{[a(1 - u_\varepsilon)](x, \xi, X)}{|\det \psi''(x, \xi, X)|_{N_{(x, \xi, X)} \text{Reg } \mathcal{C}}|^{1/2}} d(\text{Reg } \mathcal{C})(x, \xi, X).$$

On the other hand, applying our previous considerations to  $I_\varepsilon(\mu)$  instead of  $I(\mu)$ , we obtain again an asymptotic expansion of the form (24) for  $I_\varepsilon(\mu)$ , where now, the first coefficient is given by a sum of integrals of the form (19) with  $a$  replaced by  $a(1 - u_\varepsilon)$ . Since the first term in the asymptotic expansion (24) is uniquely determined, the two expressions for  $L_0(\varepsilon)$  must be identical. The statement of the lemma now follows by the Lebesgue theorem on bounded convergence.  $\square$

Note that existence of the limit in (23) has been established by partially resolving the singularities of the critical set  $\mathcal{C}$ , the corresponding limit being given by  $L_0$ . Let now  $a^+ \in C_c^\infty(\mathbb{R}^{2n} \times \mathfrak{g}, \mathbb{R}^+)$ . Since  $|u_\varepsilon| \leq 1$ , the lemma of Fatou implies that

$$\int_{\text{Reg } \mathcal{C}} \lim_{\varepsilon \rightarrow 0} \frac{[a^+(1 - u_\varepsilon)](x, \xi, X)}{|\det \psi''(x, \xi, X)|_{N_{(x, \xi, X)} \text{Reg } \mathcal{C}}|^{1/2}} d(\text{Reg } \mathcal{C})(x, \xi, X)$$

is majorized by the limit (23), with  $a$  replaced by  $a^+$ . Lemma 4 then implies that

$$\int_{\text{Reg } \mathcal{C}} \frac{a^+(x, \xi, X)}{|\det \psi''(x, \xi, X)|_{N_{(x, \xi, X)} \text{Reg } \mathcal{C}}|^{1/2}} d(\text{Reg } \mathcal{C})(x, \xi, X) < \infty.$$

Choosing now  $a^+$  to be equal 1 on the compact set  $K$  in which  $a$  was supported, and applying the theorem of Lebesgue on bounded convergence to the limit (23), we obtain Equation (21).  $\square$

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