EQUIVARIANT HEAT ASYMPTOTICS ON SPACES OF AUTOMORPHIC FORMS

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ABSTRACT. Let G be a connected, real, semisimple Lie group with finite center, and K a maximal compact subgroup of G. In this paper, we derive K-equivariant asymptotics for heat traces with remainder estimates on compact Riemannian manifolds carrying a transitive and isometric G-action. In particular, we compute the leading coefficient in the Minakshishundaram-Pleijel expansion of the heat trace for Bochner-Laplace operators on homogeneous vector bundles over compact locally symmetric spaces of arbitrary rank.

CONTENTS

1. INTRODUCTION

Let G be a connected, real, semisimple Lie group with finite center, acting isometrically and transitively on a compact, n-dimensional, real-analytic Riemannian manifold M. Let further K be a maximal compact subgroup of G. In this paper, we derive K-equivariant asymptotics for traces of heat semigroups associated to strongly elliptic operators on M with remainder estimates. In particular, if $M = \Gamma\backslash G$, where Γ is a discrete, torsion-free, uniform subgroup of G, we compute the leading coefficient in the Minakshishundaram-Pleijel expansion of the heat trace of Bochner-Laplace operators on homogeneous vector bundles over compact, locally symmetric spaces of arbitrary rank, together with an estimate for the remainder.

The study of the asymptotic behavior of heat semigroups and their kernels has a long history. One of the pioneering works in this direction was the derivation of an asymptotic expansion for the fundamental solution of the heat equation on a compact manifold by Minakshisundaram and Pleijel [15]. The first three coefficients in this expansion were computed by McKean and Singer [13] in terms of geometric quantities, yielding corresponding expansions of heat traces. This culminated in a heat theoretic proof of the index theorem by Atiyah, Bott and Patodi [1]. In the case of

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Riemannian symmetric spaces, an explicit expression for the fundamental solution of the heat equation was given by Gangolli [10] using Harish-Chandra's Plancherel theorem. Later, Donelly [9] generalized the constructions in [15] and [3] to Riemannian manifolds admitting a properly discontinuous group of isometries with compact quotient. Following these developments, Miatello [14], and DeGeorge and Wallach [8] established asymptotic expansions for heat traces of Bochner-Laplace operators on homogeneous vector bundles over compact, locally symmetric spaces of rank one. Holomorphic semigroups generated by strongly elliptic operators on Lie groups have been studied sytematically by Langlands [12], and Robinson and ter Elst [22], [24], giving lower and upper bounds for their kernels. For further references, see also [7] and [4].

To illustrate our results, let $(\pi, \mathrm{L}^2(M))$ be the regular representation of G on the Hilbert space of square integrable functions on M with respect to an invariant density, and f_t the group kernel of a strongly elliptic operator Ω of order q associated to the representation π, where $t > 0$. The corresponding heat operator is then given by $e^{-t\overline{\Omega}} = \pi(f_t)$, and characterized in Theorem 1 as a pseudodifferential operator of order $-\infty$. Due to the compactness of M, this implies that $\pi(f_t)$ is of trace class. Using this characterization, we consider the decomposition

$$\mathrm{L}^2(M) = \bigoplus_{\sigma \in \widehat{K}} \mathrm{L}^2(M)_\sigma$$

of $\mathrm{L}^2(M)$ into K-isotypic components, and derive asymptotics with remainder estimates for the trace

$$\mathrm{tr}(P_\sigma \circ \pi(f_t) \circ P_\sigma)$$

of the restriction of $\pi(f_t)$ to the isotypic component $\mathrm{L}^2(M)_\sigma = P_\sigma(\mathrm{L}^2(M))$ as t goes to zero, P_σ being the corresponding projector, see Theorem 4. In order to do so, one has to describe the asymptotic behavior of certain oscillatory integrals, which has been determined before in [21] while studying the spectrum of an invariant elliptic operator. The difficulty here resides in the fact that, since the critical sets of the corresponding phase functions are not smooth, a desingularization procedure is required in order to apply the method of the stationary phase in a suitable resolution space. In case that f_t has an asymptotic expansion of the form

$$f_t(g) \sim \frac{1}{t^{d/q}} e^{-b\left(\frac{d(g,e)^q}{t}\right)^{1/(q-1)}} \sum_{j=0}^{\infty} c_j(g) t^j, \qquad b > 0,$$

near the identity $e \in G$ with analytic coefficients $c_j(g)$, where $d = \dim G$, and $d(g,e)$ denotes the distance of $g \in G$ from the identity with respect to the canonical left-invariant metric on G, we show in Corollary 4 that

$$\mathrm{tr}(P_\sigma \circ \pi(f_t) \circ P_\sigma) = \frac{d_{\sigma\otimes\sigma}[(\pi_\sigma \otimes \pi_\sigma)_{|\mathbb{H}} : \mathbf{1}]}{(2\pi)^{n-\kappa} t^{(n-\kappa)/q}} c_0(e) \widetilde{\mathrm{vol}}\,(\Xi/\mathbb{K}) + O(t^{-(n-\kappa-1)/q} (\log t)^{\Lambda - 1}),$$

where $(\pi_\sigma, V_\sigma) \in \sigma$, and $\widetilde{\mathrm{vol}}\,(\Xi/\mathbb{K})$ is given by local integrals over the zero level set $\Xi = \mathbb{J}^{-1}(0)$ of the momentum map $\mathbb{J} : T^*M \to (\mathfrak{k} \oplus \mathfrak{k})^*$ of the underlying action of $\mathbb{K} = K \times K$ on M. In fact, $\widetilde{\mathrm{vol}}\,(\Xi/\mathbb{K})$ represents a Gaussian volume of the symplectic quotient $\Xi/\mathbb{K}$. Further, κ denotes the dimension of a K-orbit of principal type, and $\mathbb{H} \subset \mathbb{K}$ a principal isotropy group, while Λ is the maximal number of elements of a totally ordered subset of the set of $\mathbb{K}$-isotropy types.

As our main application, we consider the case $M = \Gamma \backslash G$, where $\Gamma \subset G$ is a discret, co-compact subgroup. The previous results, combined with Selberg's trace formula, then yield an asymptotic description of $L_\sigma f_t$ at the identity, where L_σ denotes the projector onto the isotypic component $\mathrm{L}^2(G)_\sigma$ of the left-regular representation $(L, \mathrm{L}^2(G))$ of G, see Proposition 3. Finally, for torsion-free Γ, we are able to compute the first coefficient in the Minakshisundaram-Pleijel expansion, together with an estimate for the remainder, of vector valued heat kernels on the compact locally symmetric space $\Gamma\backslash G/K$, generalizing part of the work in [14] and [8] to arbitrary rank. More precisely, let Δ_σ be the Bochner-Laplace operator on the homogeneous vector bundle $E_\sigma = \Gamma\backslash(G\times V_\sigma)/K \to \Gamma\backslash G/K$. Denote by λ_σ the Casimir eigenvalue of K corresponding to $\sigma\in\widehat{K}$. Then, by Theorem 5,

$$\operatorname{tr} e^{-t\Delta_\sigma} = \frac{e^{t\lambda_\sigma}\int_{\mathbb{H}} \operatorname{tr}\pi_\sigma(kk_1^{-1})\,dk_1\,dk}{(2\pi)^{\dim G/K}t^{\frac{\dim G/K}{2}}}\widetilde{\operatorname{vol}}\,(\Xi/\mathbb{K}) + O(e^{t\lambda_\sigma}t^{-(\dim G/K-1)/2}(\log t)^{\Lambda-1}),$$

where, again, $\widetilde{\operatorname{vol}}\,(\Xi/\mathbb{K})$ is given by a Gaussian volume of the symplectic quotient $\Xi/\mathbb{K}$.

This paper is organized as follows. The microlocal structure of general convolution operators with rapidly decaying group kernels on paracompact, smooth manifolds is described in Section 2. In Section 3, the Langlands kernel of a semigroup generated by a strongly elliptic operator on M is considered, and its equivariant heat trace is expressed in terms of oscillatory integrals. Since the occuring phase functions do have singular critical sets, the stationary phase principle cannot immediately be applied to describe the asymptotic behavior of those integrals. Instead, we rely on the results in [21], where resolution of singularities was used to partially resolve the singularities of the considered critical sets. This yields short-time asymptotics with remainder estimates for equivariant heat traces in Section 4. Finally, in Section 5, we consider the particular case $M = \Gamma\backslash G$, where Γ denotes a uniform, torsion-free lattice in G, and apply our results to heat traces of Bochner-Laplace operators on compact locally symmetric spaces of arbitrary rank.

2. Convolution operators

Let G be a connected, real, semisimple Lie group with finite center, and Lie algebra $\mathfrak{g}$. Denote by $\langle X,Y\rangle = \operatorname{tr}(\operatorname{ad}X\circ\operatorname{ad}Y)$ the Cartan-Killing form on $\mathfrak{g}$, and by θ a Cartan involution of $\mathfrak{g}$. Let

$$\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p} \tag{1}$$

be the Cartan decomposition of $\mathfrak{g}$ into the eigenspaces of θ, corresponding to the eigenvalues $+1$ and -1 , respectively. Put $\langle X,Y\rangle_\theta := -\langle X,\theta Y\rangle$. Then $\langle\cdot,\cdot\rangle_\theta$ defines a left-invariant metric on G. With respect to this metric, we define $d(g,h)$ as the geodesic distance between two points $g,h\in G$, and set $|g| = d(g,e)$, where e is the identity element of G. Note that $d(g_1g,g_1h) = d(g,h)$ for all $g,g_1,h\in G$. In contrast to the Killing form, $\langle\cdot,\cdot\rangle_\theta$ is no longer $\operatorname{Ad}(G)$-invariant, but still $\operatorname{Ad}(K)$-invariant, so that $d(gk,hk) = d(g,h)$ for all $g,h\in G$, and $k\in K$. Indeed, one has the following

Proposition 1. *The modified Killing form $\langle\cdot,\cdot\rangle_\theta$ is $\operatorname{Ad}(K)$-invariant, which implies that the corresponding Riemannian distance d on G is right K-invariant. In particular, $|g| = |kgk^{-1}|$ for all $g\in G$ and $k\in K$.*

Proof. This seems to be a well-known fact, but for lack of references, we include a proof here. Thus, let us first note that for $k\in K$, the morphisms $\operatorname{Ad}(k)$ and θ commute. Indeed, the inclusions $[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k}$, $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$, and $[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}$, together with the relation $\operatorname{Ad}(e^X) = e^{\operatorname{ad}X}$, $X\in\mathfrak{g}$, imply that $\operatorname{Ad}(K)\,\mathfrak{k}\subset\mathfrak{k}$, $\operatorname{Ad}(K)\,\mathfrak{p}\subset\mathfrak{p}$. Hence, $\operatorname{Ad}(k)\theta X = \theta\operatorname{Ad}(k)X$ for all $X\in\mathfrak{g}$. But then

$$\langle\operatorname{Ad}(k)X,\operatorname{Ad}(k)Y\rangle_\theta = -\langle\operatorname{Ad}(k)X,\theta\operatorname{Ad}(k)Y\rangle = -\langle\operatorname{Ad}(k)X,\operatorname{Ad}(k)\theta Y\rangle = -\langle X,\theta Y\rangle = \langle X,Y\rangle_\theta$$

for all $X,Y\in\mathfrak{g}$, $k\in K$, showing the $\operatorname{Ad}(K)$-invariance of $\langle\cdot,\cdot\rangle_\theta$.

Next, we show that the Riemannian distance d is right K-invariant. For this purpose, recall that for a curve $c:[a,b]\to\mathbf{X}$ on a Riemannian manifold $\mathbf{X}$ with metric ν, the length of c is given by

$$L(c) = \int_a^b\sqrt{\nu_{c(s)}(c'(s),c'(s))}ds.$$

Let now $c:[a,b]\to G$ be a curve in G joining two points $g,h\in G$. We then assert that

$$\frac{d}{dt}kc(t)k^{-1}|_{t=t_0} = \big(dL_{kc(t_0)k^{-1}}\big)_e \operatorname{Ad}(k)\Big((dL_{c(t_0)^{-1}})_{c(t_0)}c'(t_0)\Big), \qquad k\in G, \tag{2}$$

where $L_g: G\to G$ corresponds to left-translation by $g\in G$, and $(dL_g)_h : T_hG\to T_{gh}G$ is its differential at $h\in G$. Indeed, if $i_k: G\to G$ denotes the interior automorphism $h\mapsto khk^{-1}$, its differential at the identity e is by definition $\operatorname{Ad}(k)=(di_k)_e:\mathfrak{g}\to\mathfrak{g}$. Furthermore, since $i_k = L_k\circ R_{k^{-1}} = R_{k^{-1}}\circ L_k$, we have the identities

$$\operatorname{Ad}(k) = (dL_k)_{k^{-1}}\circ(dR_{k^{-1}})_e = (dR_{k^{-1}})_k\circ(dL_k)_e.$$

Similarly, $L_{kc(t_0)k^{-1}} = L_k\circ L_{c(t_0)}\circ L_{k^{-1}}$ implies $(dL_{kc(t_0)k^{-1}})_e = (dL_k)_{c(t_0)k^{-1}}\circ(dL_{c(t_0)})_{k^{-1}}\circ(dL_{k^{-1}})_e$. The left hand side of (2) now reads

$$\frac{d}{dt}kc(t)k^{-1}|_{t=t_0} = (dL_k)_{c(t_0)k^{-1}}\circ(dR_{k^{-1}})_{c(t_0)}(c'(t_0)),$$

while the right hand side equals

$$\begin{aligned}(dL_k)_{c(t_0)k^{-1}}&\circ(dL_{c(t_0)})_{k^{-1}}\circ(dL_{k^{-1}})_e\circ(dL_k)_{k^{-1}}\circ(dR_{k^{-1}})_e\circ(dL_{c(t_0)^{-1}})_{c(t_0)}(c'(t_0))\\ &= (dL_k)_{c(t_0)k^{-1}}\circ(dR_{k^{-1}})_{c(t_0)}\circ(dL_{c(t_0)})_e\circ(dL_{c(t_0)^{-1}})_{c(t_0)}(c'(t_0))\\ &= (dL_k)_{c(t_0)k^{-1}}\circ(dR_{k^{-1}})_{c(t_0)}(c'(t_0)),\end{aligned}$$

prooving (2). Write $\langle X,X\rangle_\theta = \|X\|_\theta^2$. The $\operatorname{Ad}(K)$-invariance of $\langle\cdot,\cdot\rangle_\theta$ then implies

$$\begin{aligned}L(kck^{-1}) &= \int_a^b \left\|(dL_{kc(s)^{-1}k^{-1}})_{kc(s)k^{-1}}\Big(\frac{d}{dt}kc(t)k^{-1}|_{t=s}\Big)\right\|_\theta ds\\ &= \int_a^b \left\|\operatorname{Ad}(k)\left[(dL_{c(t_0)^{-1}})_{c(t_0)}c'(t_0)\right]\right\|_\theta ds = \int_a^b \left\|(dL_{c(t_0)^{-1}})_{c(t_0)}c'(t_0)\right\|_\theta ds = L(c)\end{aligned}$$

for arbitrary $k\in K$. Assume now that c is a shortest geodesic. The last equality then shows that kck^{-1} is a shortest geodesic, too. Otherwise there would exist a geodesic $\tilde c$ joining kgk^{-1} and khk^{-1} with $L(\tilde c) < L(kck^{-1})$. But then $L(k^{-1}\tilde c k) < L(c)$, a contradiction. Therefore

$$d(g,h) = L(c) = L(kck^{-1}) = d(kgk^{-1},khk^{-1}) = d(gk^{-1},hk^{-1})$$

for all $g,h\in G$, $k\in K$, and the proposition follows. □

Let us consider next a paracompact C^∞-manifold M of dimension n, and assume that G acts on M in a smooth and transitive way. Let $\mathrm{C}(M)$ be the Banach space of continuous, bounded, complex valued functions on M, equipped with the supremum norm, and let $(\pi,\mathrm{C}(M))$ be the corresponding continuous regular representation of G given by

$$\pi(g)\varphi(p) = \varphi(g\cdot p), \qquad \varphi\in\mathrm{C}(M),\quad g\in G,\quad p\in M.$$

The representation of the universal enveloping algebra $\mathfrak{U}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ on the space of differentiable vectors $\mathrm{C}(M)_\infty$ will be denoted by $d\pi$. We shall also consider the regular representation of G on $\mathrm{C}^\infty(M)$ which, equipped with the topology of uniform convergence on compacta, becomes a Fréchet space. This representation will be denoted by π as well. Let $(L,\mathrm{C}^\infty(G))$ and $(R,\mathrm{C}^\infty(G))$ be the left, respectively right regular representation of G. A function f on G is said to be of *at most of exponential growth*, if there exists a $\kappa>0$ such that $|f(g)|\leq Ce^{\kappa|g|}$ for some constant $C>0$, and all $g\in G$. Let dg be a Haar measure on G. We then make the following

Definition 1. *The space of rapidly decreasing functions on G, denoted by $\mathcal{S}(G)$, is given by all functions $f\in\mathrm{C}^\infty(G)$ satisfying the following conditions:*

i) *For every* $\kappa \geq 0$*, and* $X \in \mathfrak{U}$*, there exists a constant* $C > 0$ *such that*

$$|dL(X)f(g)| \leq Ce^{-\kappa|g|};$$

ii) *for every* $\kappa \geq 0$*, and* $X \in \mathfrak{U}$*, one has* $dL(X)f \in \mathrm{L}^1(G, e^{\kappa|g|}d_G)$*.*

The space $\mathcal{S}(G)$ was first introduced in [20], and motivated by the study of strongly elliptic operators, and the semigroups generated by them, see Section 3. Let us now associate to every $f \in \mathcal{S}(G)$ and $\varphi \in \mathrm{C}(M)$ the vector-valued integral $\int_G f(g)\pi(g)\varphi \, d_G(g) \in \mathrm{C}(M)$, yielding a continuous linear operator

$$\pi(f) = \int_G f(g)\pi(g)\, dg \tag{3}$$

on $\mathrm{C}(M)$. Its restriction to $\mathrm{C}^\infty_\mathrm{c}(M)$ induces a continuous linear operator

$$\pi(f) : \mathrm{C}^\infty_\mathrm{c}(M) \longrightarrow \mathrm{C}(M) \subset \mathcal{D}'(M),$$

with Schwartz kernel given by the distribution section $\mathcal{K}_f \in \mathcal{D}'(M \times M, \mathbf{1} \boxtimes \Omega_M)$, where Ω_M denotes the density bundle of M. In what follows, we shall show that $\pi(f)$ is an operator with smooth kernel. As we shall see, the smoothness of the operators $\pi(f)$ is a direct consequence of the fact that G acts transitively on M.

Thus, let $\left\{(\widetilde{W}'_\iota, \varphi_\iota)\right\}_{\iota \in I}$ be a locally finite atlas of M. By [11], page 273, there exists a locally finite refinement $\left\{\widetilde{W}_\iota\right\}_{\iota \in I}$ with the same index set such that $\overline{\widetilde{W}_\iota} \subset \widetilde{W}'_\iota$ for every $\iota \in I$. Assume that the $\overline{\widetilde{W}'_\iota}$ are compact, and let $\{\alpha_\iota\}_{\iota \in I}$ be a partition of unity subordinated to the atlas $\left\{(\widetilde{W}_\iota, \varphi_\iota)\right\}_{\iota \in I}$, meaning that

(a) the α_ι are smooth functions, and $0 \leq \alpha_\iota \leq 1$;
(b) $\operatorname{supp} \alpha_\iota \subset \widetilde{W}_\iota$;
(c) $\sum_{\iota \in I} \alpha_\iota = 1$.

Let further $\{\alpha'_\iota\}_{\iota \in I}$ be another set of functions satisfying condition (a), and in addition

(b') $\operatorname{supp} \alpha'_\iota \subset \widetilde{W}'_\iota$;
(c') $\alpha'_{\iota|\widetilde{W}_\iota} \equiv 1$.

Consider now the localization of $\pi(f)$ with respect to the latter atlas

$$A^\iota_f u = [\pi(f)_{|\widetilde{W}_\iota}(u \circ \varphi_\iota)] \circ \varphi_\iota^{-1}, \qquad u \in \mathrm{C}^\infty_\mathrm{c}(W_\iota),\, W_\iota = \varphi_\iota(\widetilde{W}_\iota) \subset \mathbb{R}^n,$$

corresponding to the diagram

$$\begin{array}{ccc}
\mathrm{C}^\infty_\mathrm{c}(\widetilde{W}_\iota) & \xrightarrow{\pi(f)_{|\widetilde{W}_\iota}} & \mathrm{C}^\infty(\widetilde{W}_\iota) \\
\varphi_\iota^* \Big\uparrow & & \Big\uparrow \varphi_\iota^* \\
\mathrm{C}^\infty_\mathrm{c}(W_\iota) & \xrightarrow{A^\iota_f} & \mathrm{C}^\infty(W_\iota).
\end{array}$$

Let $p \in \widetilde{W}_\iota$. Writing $\varphi^g_\iota = \varphi_\iota \circ g \circ \varphi_\iota^{-1}$, and $x = \varphi_\iota(p) = (x_1, \dots, x_n) \in W_\iota$ we obtain

$$A^\iota_f u(x) = \int_G f(g)[(u \circ \varphi_\iota)\alpha'_\iota](g \cdot \varphi_\iota^{-1}(x))\; dg = \int_G f(g)c_\iota(x,g)(u \circ \varphi^g_\iota)(x)\, dg,$$

where we put $c_\iota(x,g) = \alpha'_\iota(g \cdot \varphi_\iota^{-1}(x))$. Next, define the functions

$$a^\iota_f(x,\xi) = e^{-ix\cdot\xi} \int_G e^{i\varphi^g_\iota(x)\cdot\xi} c_\iota(x,g) f(g)\, dg. \tag{4}$$

Since f is rapidly falling, differentiation under the integral yields $a^\iota_f(x,\xi) \in \mathrm{C}^\infty(W_\iota \times \mathbb{R}^n)$. We can now state

Theorem 1 (Structure theorem)**.** *Let M be a paracompact C^∞-manifold of dimension n, and G a connected, real, semisimple Lie group with finite center acting on M in a smooth and transitive way. Let further $f \in \mathcal{S}(G)$ be a rapidly decaying function on G. Then the operator $\pi(f)$ is a pseudodifferential operator of class $\mathrm{L}^{-\infty}(M)$, that is, it is locally of the form* [1]

$$A^\iota_f u(x) = \int e^{ix\cdot\xi} a^\iota_f(x,\xi)\hat{u}(\xi)\, d\!\!{}^-\xi, \qquad u \in \mathrm{C}^\infty_{\mathrm{c}}(W_\iota), \tag{5}$$

where the symbol $a^\iota_f(x,\xi) \in \mathrm{S}^{-\infty}(W_\iota, \mathbb{R}^n)$ is given by (4), and $d\!\!{}^-\xi = (2\pi)^{-n}\, d\xi$. In particular, the kernel of the operator A^ι_f is given by the oscillatory integral

$$K_{A^\iota_f}(x,y) = \int e^{i(x-y)\cdot\xi} a^\iota_f(x,\xi)\, d\!\!{}^-\xi \in \mathrm{C}^\infty(W_\iota \times W_\iota). \tag{6}$$

Proof. Our considerations will essentially follow the proof of Theorem 4 in [20], or Theorem 2 in [19]. For a review on pseudodifferential operators, the reader is referred to [23]. Fix a chart $(\widetilde{W}_\iota, \varphi_\iota)$, and let $p \in \widetilde{W}_\iota$, $x = (x_1, \dots, x_n) = \varphi_\iota(p) \in \mathbb{R}^n$. In what follows we shall show that $a^\iota_f(x,\xi)$ belongs to the symbol class $\mathcal{S}^{-\infty}(W_\iota \times \mathbb{R}^n)$. For later purposes, we shall actually consider the slightly more general amplitudes

$$\begin{aligned} a^{\iota\tilde{\iota}}_f(x,\xi;k_1,k_2) &= e^{-i\varphi^{k_1k_2}_{\tilde{\iota}}(x)\cdot\xi}\alpha'_{\tilde{\iota}}(k_1k_2\cdot\varphi^{-1}_\iota(x)) \int_G e^{i\varphi^{k_1gk_2}_\iota(x)\cdot\xi} c_\iota(x, k_1gk_2) f(g)\, dg \\ &= e^{-i\varphi^{k_1k_2}_{\tilde{\iota}}(x)\cdot\xi}\alpha'_{\tilde{\iota}}(k_1k_2\cdot\varphi^{-1}_\iota(x)) \int_G e^{i\varphi^{g}_\iota(x)\cdot\xi} c_\iota(x, g)(L(k_1)R(k_2^{-1})f)(g)\, dg, \end{aligned} \tag{7}$$

where $k_1, k_2 \in G$. Here we took into account the unimodularity of G. In particular, $a^\iota_f(x,\xi) = a^{\iota\iota}_f(x,\xi;e,e)$. Denote by $V_{\iota,p}$ the set of all $g \in G$ such that $g\cdot p \in \widetilde{W}_\iota$. Assume that $g \in V_{\iota,p}$, and write

$$\psi^\iota_{\xi,x}(g) = e^{i\varphi^g_\iota(x)\cdot\xi}.$$

For $X \in \mathfrak{g}$ one computes that

$$dL(X)\psi^\iota_{\xi,x}(g) = \frac{d}{ds} e^{i\varphi^{\mathrm{e}^{-sX}g}_\iota(x)\cdot\xi}{}_{|s=0} = i\psi^\iota_{\xi,x}(g)\sum_{i=1}^n \xi_i dL(X)x_{i,p}(g),$$

where we put $x_{i,p}(g) = x_i(g\cdot p)$. Let $\{X_1, \dots, X_d\}$ be a basis of $\mathfrak{g}$. Since G acts locally transitively on $\widetilde{W}_\iota$, the $n \times d$ matrix

$$(dL(X_j)x_{i,p}(g))_{i,j}$$

has maximal rank. As a consequence, there exists a neighborhood $\widetilde{U}_p$ of p, and indices $j_1, \dots, j_n$ such that

$$\det\, (dL(X_{j_k})x_{i,p'}(g))_{i,k} \neq 0 \qquad \forall p' \in \widetilde{U}_p.$$

Hence,

$$\begin{pmatrix} dL(X_{j_1})\psi^\iota_{\xi,x'}(g) \\ \vdots \\ dL(X_{j_n})\psi^\iota_{\xi,x'}(g) \end{pmatrix} = i\psi^\iota_{\xi,x'}(g)\mathcal{M}(x',g)\xi, \tag{8}$$

[1]Here and in what follows we use the convention that, if not specified otherwise, integration is to be performed over whole Euclidean space.

where $\mathcal{M}(x',g) = \Big(dL(X_{j_k})x_{i,\varphi_\iota^{-1}(x')}(g)\Big)_{i,k} \in \mathrm{GL}(n,\mathbb{R})$ is an invertible matrix for all $x' \in \varphi_\iota(\widetilde{U}_p)$. Consider now the extension of $\mathcal{M}(x',g)$ as an endomorphism in $\mathbb{C}^1[\mathbb{R}^n_\xi]$ to the symmetric algebra $\mathrm{S}(\mathbb{C}^1[\mathbb{R}^n_\xi]) \simeq \mathbb{C}[\mathbb{R}^n_\xi]$. Since $\mathcal{M}(x',g)$ is invertible, its extension to $\mathrm{S}^N(\mathbb{C}^1[\mathbb{R}^n_\xi])$ is also an automorphism for any $N \in \mathbb{N}$. Regarding the polynomials $\xi_1,\dots,\xi_n$ as a basis in $\mathbb{C}^1[\mathbb{R}^n_\xi]$, let us denote the image of the basis vector ξ_j under the endomorphism $\mathcal{M}(x',g)$ by $\mathcal{M}\xi_j$, so that by (8)

$$\mathcal{M}\xi_k = -i\psi^\iota_{-\xi,x'}(g)dL(X_{j_k})\psi^\iota_{\xi,x'}(g), \qquad 1 \le k \le n.$$

In this way, each polynomial $\xi_{j_1} \otimes \dots \otimes \xi_{j_N} \equiv \xi_{j_1}\dots\xi_{j_N}$ can be written as a linear combination

$$\xi^\alpha = \sum_\beta \Lambda^\alpha_\beta(x',g)\mathcal{M}\xi_{\beta_1}\cdots\mathcal{M}\xi_{\beta_{|\alpha|}}, \tag{9}$$

where the $\Lambda^\alpha_\beta(x',g)$ are smooth functions given in terms of the matrix coefficients of $\mathcal{M}(x',g)$. We now have for arbitrary indices $\beta_1,\dots,\beta_r$ and all $x' \in \varphi_\iota(\widetilde{U}_p)$

$$\begin{aligned} i^r\psi^\iota_{\xi,x'}(g)\mathcal{M}\xi_{\beta_1}\cdots\mathcal{M}\xi_{\beta_r} &= dL(X_{\beta_1}\cdots X_{\beta_r})\psi^\iota_{\xi,x'}(g) \\ &\quad + \sum_{s=1}^{r-1}\sum_{\alpha_1,\dots,\alpha_s} d^{\beta_1,\dots,\beta_r}_{\alpha_1,\dots,\alpha_s}(x',g)dL(X_{\alpha_1}\cdots X_{\alpha_s})\psi^\iota_{\xi,x'}(g), \end{aligned} \tag{10}$$

where the coefficients $d^{\beta_1,\dots,\beta_r}_{\alpha_1,\dots,\alpha_s}(x',g)$ are smooth functions given by the matrix coefficients of $\mathcal{M}(x',g)$ which are at most of exponential growth in g, and independent of ξ, see Lemma 4 in [19]. The key step in proving the theorem is that, as an immediate consequence of equations (9) and (10), we can express $(1+|\xi|^2)^N$ as a linear combination of derivatives $dL(X^\alpha)\psi^\iota_{\xi,x'}(g)$, obtaining for arbitrary $N \in \mathbb{N}$ and $x' \in \varphi_\iota(\widetilde{U}_p)$ the equality

$$\psi^\iota_{\xi,x'}(g)(1+|\xi|^2)^N = \sum_{r=0}^{2N}\sum_{|\alpha|=r} b^N_\alpha(x',g)dL(X^\alpha)\psi^\iota_{\xi,x'}(g), \tag{11}$$

where the coefficients $b^N_\alpha(x',g)$ are at most of exponential growth in g. Let us now show that $a^{\iota\tilde\iota}_f(x,\xi;k_1,k_2) \in \mathrm{S}^{-\infty}(W_\iota \times \mathbb{R}^n_\xi)$ for each fixed $k_1,k_2 \in K$. Note that $a^{\iota\tilde\iota}_f(x,\xi;k_1,k_2) \in \mathrm{C}^\infty(W_\iota \times \mathbb{R}^n_\xi \times K \times K)$. While differentiation with respect to ξ does not alter the growth properties of the functions $a^{\iota\tilde\iota}_f(x,\xi;k_1,k_2)$, differentiation with respect to x yields additional powers in ξ. As one computes, $(\partial^\alpha_\xi\,\partial^\beta_x\,a^{\iota\tilde\iota}_f)(x,\xi;k_1,k_2)$ is a finite sum of terms of the form

$$\xi^\delta e^{-i\varphi^{k_1k_2}_\iota(x)\cdot\xi}\int_G \psi^\iota_{\xi,x}(g)(L(k_1)R(k_2^{-1})f)(g)d^\delta_{\beta'\beta''}(x,k_1,k_2,g)(\partial^{\beta'}_x\,c_\iota)(x,g)\,\partial^{\beta''}_x[\alpha'_{\tilde\iota}(k_1k_2\cdot\varphi_\iota^{-1}(x))]\,dg,$$

the functions $d^\delta_{\beta'\beta''}(x,k_1,k_2,g)$ being at most of exponential growth in g. Let next $f_1 \in \mathcal{S}(G)$, and assume that $f_2 \in \mathrm{C}^\infty(G)$, together with all its derivatives, is at most of exponential growth. Then, by [20], Proposition 1, we have

$$\int_G f_1(g)dL(X^\iota)f_2(g)d_G(g) = (-1)^{|\iota|}\int_G dL(X^{\tilde\iota})f_1(g)f_2(g)d_G(g), \tag{12}$$

where for $X^\iota = X^{\iota_1}_{i_1}\dots X^{\iota_r}_{i_r}$ we wrote $X^{\tilde\iota} = X^{\iota_r}_{i_r}\dots X^{\iota_1}_{i_1}$, ι being an arbitrary multi-index. Let now $\mathcal{O}$ denote an arbitrary compact set in W_ι. By Heine–Borel, $\varphi_\iota^{-1}(\mathcal{O})$ can be covered by a finite number of neighborhoods $\widetilde{U}_p$. Making use of equation (11), and integrating according to (12), we obtain for arbitrary multi-indices α,β the estimate

$$|(\partial^\alpha_\xi\,\partial^\beta_x\,a^{\iota\tilde\iota}_f)(x,\xi;k_1,k_2)| \le \frac{1}{(1+\xi^2)^N}C_{\alpha,\beta,\mathcal{O}} \qquad x \in \mathcal{O},$$

where $N \in \mathbb{N}$, since $L(k_1)R(k_2^{-1})f \in \mathcal{S}(G)$. This proves that $a_f^{\iota\tilde{\iota}}(x,\xi;k_1,k_2) \in \mathrm{S}^{-\infty}(W_\iota \times \mathbb{R}^n_\xi)$ for each fixed $k_1, k_2 \in K$. Since equation (5) is an immediate consequence of the Fourier inversion formula, the proof of the theorem is now complete. □

Let dM be a fixed G-invariant density on M, and denote by $\mathrm{L}^2(M)$ the space of square integrable functions on M. In case that M is compact, the fact that the integral operators $\pi(f)$ have smooth kernels implies that they are trace-class operators in $\mathrm{L}^2(M)$. Indeed, one has the following

Lemma 1. *Let $\mathbf{X}$ be a compact manifold of dimension n with volume form $d\,\mathbf{X}$. Let $k: \mathbf{X}\times\mathbf{X} \to \mathbb{C}$ be a kernel function of class $\mathrm{C}^{(n+1)}(\mathbf{X}\times\mathbf{X})$. Then the operator*

$$(Kf)(p) = \int_{\mathbf{X}} k(p,q)f(q)d\,\mathbf{X}(q), \qquad f \in L^2(\mathbf{X}, d\,\mathbf{X}),$$

is trace class, and $\operatorname{tr} K = \int_{\mathbf{X}} k(p,p)\, d\,\mathbf{X}(p)$.

Proof. See [14], Lemma 2.2. □

In our situation, we obtain

Corollary 1. *Let M be a compact, C^∞-manifold of dimension n, and G a connected, real, semisimple Lie group with finite center acting on M in a transitive way. If $f \in \mathcal{S}(G)$, then $\pi(f)$ is a trace class operator in $\mathrm{L}^2(M)$, and*

$$\operatorname{tr}\pi(f) = \sum_\iota \int_{W_\iota} (\alpha_\iota \circ \varphi_\iota^{-1})(x) K_{A_f^\iota}(x,x)\, dx = \sum_\iota \int_M \alpha_\iota(p) K_{A_f^\iota}(\varphi_\iota(p), \varphi_\iota(p)) j_\iota(p) dM(p), \tag{13}$$

where dx denotes Lebesgue measure in $\mathbb{R}^n$, and $(\varphi_\iota)^*(dx) = j_\iota dM$.

Proof. By Theorem 1 , $\mathcal{K}_f \in \mathrm{C}^\infty(M\times M, \mathbf{1}\boxtimes\Omega_M)$. Locally, the kernel $\mathcal{K}_f$ is determined by the smooth functions (6). Restricting the latter to the respective diagonals in W_ι, one obtains a family of functions on M

$$k_f^\iota(p) = K_{A_f^\iota}(\varphi_\iota(p), \varphi_\iota(p)), \qquad p \in \widetilde{W}_\iota,$$

which define a density $k_f dM \in \mathrm{C}^\infty(M, \Omega_M)$ on M. Since M is compact, it can be integrated, and by Lemma 1 we get

$$\operatorname{tr}\pi(f) = \int_M k_f(p)\, dM(p) = \sum_\iota \int_{W_\iota} (\alpha_\iota \circ \varphi_\iota^{-1})(x) K_{A_f^\iota}(x,x)\, dx = \sum_\iota \int_{\widetilde{W}_\iota} \alpha_\iota(p) k_f^\iota(p) j_\iota(p) dM(p),$$

where we wrote $(\varphi_\iota)^*(dx) = j_\iota dM$. □

3. Equivariant heat asymptotics

From now on, let M be a closed, real-analytic Riemannian manifold of dimension n, and G a connected, real, semisimple Lie group with finite center acting transitively and isometrically on M. Assume that M is endowed with a G-invariant density dM. Consider further a maximal compact subgroup K of G, and let $\widehat{K}$ denote the set of all equivalence classes of unitary irreducible representations of K. Let (π_σ, V_σ) be a unitary irreducible representation of K of dimension d_σ belonging to $\sigma \in \widehat{K}$, and $\chi_\sigma(k) = \operatorname{tr}\pi_\sigma(k)$ the corresponding character. As a unitary representation of K, $(\pi, \mathrm{L}^2(M))$ decomposes into isotypic components according to

$$\mathrm{L}^2(M) \simeq \bigoplus_{\sigma\in\widehat{K}} \mathrm{L}^2(M)_\sigma,$$

where $\mathrm{L}^2(M)_\sigma = P_\sigma(\mathrm{L}^2(M))$, and $P_\sigma = d_\sigma \int_K \overline{\chi_\sigma(k)}\pi(k)\, dk$ is the corresponding projector in $\mathrm{L}^2(M)$, dk being a Haar measure on K. Let $f \in \mathcal{S}(G)$, and consider the restriction $P_\sigma \circ \pi(f) \circ P_\sigma$ of the integral operator $\pi(f)$ to the isotypic component $\mathrm{L}^2(M)_\sigma$. As one computes, for $\varphi \in \mathrm{L}^2(M)$,

$$\begin{aligned}[P_\sigma \circ \pi(f) \circ P_\sigma]\varphi(p) &= d_\sigma^2 \int_K \overline{\chi_\sigma(k)}[\pi(f) \circ P_\sigma]\varphi(k \cdot p)\, dk \\ &= d_\sigma^2 \int_K \int_G \overline{\chi_\sigma(k)} f(g) P_\sigma \varphi(gk \cdot p)\, dg\, dk \\ &= d_\sigma^2 \int_K \int_G \int_K \overline{\chi_\sigma(k)} f(g) \overline{\chi_\sigma(k_1)} \varphi(k_1 g k \cdot p)\, dk_1\, dg\, dk.\end{aligned}$$

Since G is unimodular, one obtains

$$P_\sigma \circ \pi(f) \circ P_\sigma = \pi(H_f^\sigma), \tag{14}$$

where $H_f^\sigma \in \mathcal{S}(G)$ is given by

$$H_f^\sigma(g) = d_\sigma^2 \int_K \int_K f(k_1^{-1} g k^{-1}) \overline{\chi_\sigma(k_1)\chi_\sigma(k)} dk\, dk_1. \tag{15}$$

Clearly, $H_f^\sigma \in \mathcal{S}(G)$, compare [2], Proposition 2.4. Note that if f is K-bi-invariant, $\pi(f)$ commutes with P_σ, so that $P_\sigma \circ \pi(f) \circ P_\sigma = P_\sigma \circ \pi(f) = \pi(f) \circ P_\sigma$. In Section 5, we shall also consider kernels of the form

$$\int_K \int_K f(k_1^{-1} g k^{-1}) \sigma_{ij}(k) \sigma_{lm}(k_1)\, dk\, dk_1$$

where $\sigma_{ij}(k) = \langle e_i, \pi_\sigma(k) e_j \rangle$ are matrix elements of σ with respect to a basis $\{e_i\}$ of V_σ. With the notation as in the previous section we now have the following

Proposition 2. *Let $f \in \mathcal{S}(G)$, and $\sigma \in \widehat{K}$. Then $\pi(H_f^\sigma)$ is of trace class, and*

$$\begin{aligned}\mathrm{tr}\ \pi(H_f^\sigma) = &\frac{d_\sigma^2}{(2\pi)^n} \sum_{\iota,\tilde{\iota}} \int_K \int_K \int_{T^*M} e^{i\Phi_{\iota\tilde{\iota}}(p,\xi,k_1,k)} \alpha_\iota(p) \alpha_{\tilde{\iota}}(k_1 k \cdot p) \overline{\chi_\sigma(k_1)\chi_\sigma(k)} \\ &\cdot a_f^{\iota\tilde{\iota}}(\varphi_\iota(p), \xi; k_1, k) j_\iota(p) d(T^*M)(p,\xi)\, dk\, dk_1,\end{aligned}$$

where $d(T^*M)(p,\xi)$ denotes the canonical density on the cotangent bundle T^*M, and we set

$$\Phi_{\iota\tilde{\iota}}(p,\xi,k_1,k) = (\varphi_{\tilde{\iota}}(k_1 k \cdot p) - \varphi_\iota(p)) \cdot \xi,$$

while $a_f^{\iota\tilde{\iota}}(x,\xi;k_1,k_2) \in \mathrm{S}^{-\infty}(W_\iota \times \mathbb{R}^n_\xi)$ was defined in (7).

Proof. By Corollary 1, $\pi(H_f^\sigma)$ is of trace class, and at the microlocal level one has

$$\Big[\pi(H_f^\sigma)(u \circ \varphi_\iota)\Big](\varphi_\iota^{-1}(x)) = A^\iota_{H_f^\sigma} u(x), \qquad u \in \mathrm{C^\infty_c}(W_\iota),$$

where $A^\iota_{H_f^\sigma}$ is given by (5). By the unimodularity of G, together with (4) and (6),

$$\begin{aligned}K_{A^\iota_{H_f^\sigma}}(x,y) &= \int e^{i(x-y)\cdot\xi} a^\iota_{H_f^\sigma}(x,\xi)\, đ\xi = \int \left[\int_G e^{i(\varphi_\iota^g(x)-y)\cdot\xi} c_\iota(x,g) H_f^\sigma(g)\, dg\right] đ\xi \\ &= d_\sigma^2 \int \left[\int_G \int_K \int_K f(g) e^{i(\varphi_\iota^{k_1 g k}(x)-y)\cdot\xi} c_\iota(x, k_1 g k) \overline{\chi_\sigma(k_1)\chi_\sigma(k)}\, dk\, dk_1\, dg\right] đ\xi.\end{aligned}$$

Let $\psi \in \mathrm{C^\infty_c}(\mathbb{R}^n, \mathbb{R}^+)$ be equal 1 near the origin, and $\varepsilon > 0$. By Lebesgue's theorem on bounded convergence,

$$K_{A^\iota_{H_f^\sigma}}(x,y) = \lim_{\varepsilon \to 0} \int e^{i(x-y)\cdot\xi} a^\iota_{H_f^\sigma}(x,\xi) \psi(\varepsilon\xi)\, đ\xi,$$

since $a^\iota_{H^\sigma_f}(x,\xi)$ is rapidly falling in ξ. Arguing as in the proof of Corollary 1, one obtains for $\operatorname{tr}\pi(H^\sigma_f)$ the expression

$$\begin{aligned}
&\lim_{\varepsilon\to 0} d_\sigma^2 \sum_\iota \int_{\widetilde{W}_\iota}\int\int_G\int_K\int_K e^{i(\varphi_\iota(k_1gk\cdot p)-\varphi_\iota(p))\cdot\xi} f(g)\alpha_\iota(p)c_\iota(\varphi_\iota(p),k_1gk)\overline{\chi_\sigma(k_1)\chi_\sigma(k)}\psi(\varepsilon\xi)\\
&\qquad \cdot j_\iota(p)\,dk\,dk_1\,dg\,đ\xi\,dM(p)\\
&= \lim_{\varepsilon\to 0} d_\sigma^2 \sum_{\iota,\tilde\iota} \int_{W_\iota}\int\int_G\int_K\int_K e^{i(\varphi_\iota(k_1gk\cdot p)-\varphi_{\tilde\iota}(k_1k\cdot p))\cdot\xi} e^{i(\varphi_{\tilde\iota}(k_1k\cdot p)-\varphi_\iota(p))\cdot\xi} f(g)\alpha_\iota(p)\\
&\qquad \cdot\alpha'_\iota(k_1gk\cdot p)\alpha_{\tilde\iota}(k_1k\cdot p)\alpha'_{\tilde\iota}(k_1k\cdot p)\overline{\chi_\sigma(k_1)\chi_\sigma(k)}\psi(\varepsilon\xi)j_\iota(p)\;dk\,dk_1\,dg đ\xi\,dM(p)\\
&= \lim_{\varepsilon\to 0} d_\sigma^2 \sum_{\iota,\tilde\iota} \int_K\int_K\int_{\widetilde{W}_\iota}\int e^{i(\varphi_{\tilde\iota}(k_1k\cdot p)-\varphi_\iota(p))\cdot\xi}\alpha_\iota(p)\alpha_{\tilde\iota}(k_1k\cdot p)\overline{\chi_\sigma(k_1)\chi_\sigma(k)}\psi(\varepsilon\xi)\\
&\qquad \cdot a^{\iota\tilde\iota}_f(\varphi_\iota(p),\xi;k_1,k)j_\iota(p)\,đ\xi\,dM(p)\,dk\,dk_1,
\end{aligned}$$

where the change of order of integration is permissible, since everything is absolutely convergent. Note that we used the equality

$$1=\sum_{\tilde\iota}\alpha_{\tilde\iota}(k_1k\cdot p)\alpha'_{\tilde\iota}(k_1k\cdot p).$$

Finally, it was shown in the proof of Theorem 1 that $a^{\iota\tilde\iota}_f(\varphi_\iota(p),\xi;k_1,k)$ is rapidly falling in ξ, so that we can pass to the limit under the integral, and the assertion follows. □

In what follows, we shall address the case where $f=f_t\in\mathcal{S}(G)$, $t>0$, is the Langlands kernel of a semigroup generated by a strongly elliptic operator associated to the representation π. Our main goal will be the derivation of asymptotics for

$$\operatorname{tr}\pi(H^\sigma_{f_t})=\operatorname{tr}(P_\sigma\circ\pi(f_t)\circ P_\sigma)$$

as $t\to 0^+$. Thus, let $\mathcal{G}$ be a Lie group and $(\pi,\mathcal{B})$ a continuous representation of $\mathcal{G}$ in some Banach space $\mathcal{B}$. Denote by $\mathfrak{g}$ the Lie algebra of $\mathcal{G}$, and by $X_1,\dots,X_d$ a basis of it. Consider further a strongly elliptic differential operator of order q associated to π

$$\Omega=\sum_{|\alpha|\le q}c_\alpha d\pi(X^\alpha), \tag{16}$$

meaning that $\operatorname{Re}(-1)^{q/2}\sum_{\alpha=q}c_\alpha\xi^\alpha\ \ge\ \kappa|\xi|^q$ for all $\xi\in\mathbb{R}^d$, and some $\kappa>0$. The general theory of strongly continuous semigroups establishes that its closure generates a strongly continuous holomorphic semigroup of bounded operators which is given by

$$S_\tau=\frac{1}{2\pi i}\int_\Lambda e^{\lambda\tau}(\lambda\mathbb{1}+\overline{\Omega})^{-1}d\lambda, \tag{17}$$

where Λ is an appropiate path in $\mathbb{C}$ coming from infinity and going to infinity, and $|\arg\tau|<\eta$ for an appropiate $\eta\in(0,\pi/2]$. The integral converges uniformly with respect to the operator norm, and for $t>0$, the semigroup S_t can be characterized by a convolution semigroup $\{\mu_t\}_{t>0}$ of complexes measures on $\mathcal{G}$ according to

$$S_t=\int_{\mathcal{G}}\pi(g)d\mu_t(g),$$

the representation π being measurable with respect to the measures μ_t. The μ_t are absolutely continuous with respect to Haar measure $d_{\mathcal{G}}$ on $\mathcal{G}$ so that, if we denote by $f_t(g)\in L^1(\mathcal{G},d_{\mathcal{G}})$ the corresponding Radon-Nikodym derivatives, one has an expressions

$$S_t = \pi(f_t) = \int_G f_t(g)\pi(g) d_{\mathcal{G}}(g), \qquad t > 0. \tag{18}$$

The function $f_t(g) \in L^1(\mathcal{G}, d_{\mathcal{G}})$ is analytic in $t \in \mathbb{R}^+_*$ and $g \in \mathcal{G}$, universal for all Banach representations, and one can show that $f_t \in \mathcal{S}(\mathcal{G})$. Moreover, it satisfies the following L^∞ upper bounds. There exist constants $a, b, c_1, c_2 > 0$ and $\omega \geq 0$ such that

$$|(dL(X^\alpha)\partial_t^l f_t)(g)|_{t=\tau} \leq a c_1^{|\alpha|} c_2^l |\alpha|!\, l!\, \tau^{-\frac{|\alpha|+d}{q}-l} e^{\omega\tau} e^{-b(|g|^q/\tau)^{1/(q-1)}} \tag{19}$$

for all $\tau > 0$, $g \in G$, $l \in \mathbb{N}$, and multi-indices α. For a complete exposition of these facts the reader is referred to [22], pages 30, 152, and 209, or [24]. In what follows, we shall call f_t the *Langlands, or group kernel* of the holomorphic semigroup S_t. Returning to our situation, let $\mathcal{G} = G$, and π be the regular representation of G on $\mathrm{L}^2(M)$. Let us mention that as a consequence of the bounds (19), we have the following

Corollary 2. *There exist constants $a, b, c_1, c_2 > 0$ and $\omega \geq 0$ such that*

$$|(dL(X^\alpha)\partial_t^l H^\sigma_{f_t})(g)|_{t=\tau} \leq a c_1^{|\alpha|} c_2^l |\alpha|!\, l!\, \tau^{-\frac{|\alpha|+d}{q}-l} e^{\omega\tau} e^{-b(\frac{d(gK,K)^q}{\tau})^{1/(q-1)}}$$

for all $\tau > 0$, $g \in G$, $l \in \mathbb{N}$, and multi-indices α.

Proof. Clearly,

$$|H^\sigma_{f_t}(g)|_{t=\tau} \leq d_\sigma^2 \int_K \int_K |f_t(k_1^{-1} g k^{-1}))|\, dk\, dk_1.$$

According to (19) we therefore have

$$|H^\sigma_{f_t}(g)| \leq d_\sigma^2 a\, t^{-\frac{d}{q}} e^{\omega t} \int_K \int_K \exp\left(-b \left(\frac{|k_1^{-1} g k^{-1}|^q}{t} \right)^{\frac{1}{q-1}} \right) dk\, dk_1,$$

where $|k_1^{-1}gk^{-1}| = d(k_1^{-1}gk^{-1}, e) = d(gk^{-1}, k_1)$. Put $\mathbb{X} = G/K$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. By restriction of the Killing form to $T_e\mathbb{X} \simeq \mathfrak{p}$ one obtains an invariant Riemannian metric on $\mathbb{X}$ such that the canonical projection map $G \to \mathbb{X}$ becomes a Riemannian submersion. Now, if $d(gK, hK)$ denotes the geodesic distance on $\mathbb{X}$,

$$|g| = d(g, e) \geq d(gK, K), \qquad g \in G,$$

compare [17], Theorem 3.1. By applying similar arguments to the derivatives, the corollary follows. □

Let $\beta \in \mathrm{C}^\infty_c(G)$, $0 \leq \beta \leq 1$ have support in a sufficiently small neighborhood U of $e \in G$ satisfying $U = U^{-1}$, and assume that $\beta = 1$ close to e. We then have the following

Theorem 2. *Consider a strongly elliptic differential operator Ω of order $q \geq 2$ associated to $(\pi, \mathrm{L}^2(M))$, and the corresponding semigroup $S_t = \pi(f_t)$ with Langlands kernel f_t, $t > 0$. Let $\sigma \in \widehat{K}$. Then*

$$\operatorname{tr} \pi(H^\sigma_{f_t}) = \operatorname{tr} \pi(H^\sigma_{f_t\beta}) + O(t^\infty),$$

where

$$\operatorname{tr} \pi(H^\sigma_{f_t\beta}) = \frac{d_\sigma^2}{(2\pi)^n t^{n/q}} \sum_\iota \int_K \int_K \int_{T^*M} e^{i\Phi_{\iota\iota}(p,\xi,k_1,k)/t^{1/q}} \alpha_\iota(p) \overline{\chi_\sigma(k_1)\chi_\sigma(k)}$$
$$\cdot\, b^\iota_{f_t}(\varphi_\iota(p), \xi/t^{1/q}; k_1, k) j_\iota(p)\, d(T^*M)(p,\xi)\, dk\, dk_1, \qquad t > 0,$$

and

$$b^\iota_{f_t}(\varphi_\iota(p), \xi; k_1, k) = e^{-i\varphi_\iota(k_1 k\cdot p)\cdot\xi} \int_U e^{i\varphi_\iota(k_1 g k\cdot p)\cdot \xi} c_\iota(\varphi_\iota(p), k_1 g k) f_t(g) \beta(g)\, dg$$

is rapidly decaying in ξ, *and vanishes if* $k_1 k \cdot p \notin \tilde{W}'_\iota$. *Furthermore, for any multi-indices* $\alpha, \beta, \delta_1, \delta$

$$|\, \partial_x^\alpha \, \partial_\xi^\beta \, \partial_{k_1}^{\delta_1} \, \partial_k^\delta [b^\iota_{f_t}(x, \xi/t^{1/q}; k_1, k)]| \leq C$$

for some constant $C > 0$ *independent of* $0 < t < 1$.

Proof. To determine the asymptotic behavior of $\operatorname{tr} \pi(H^\sigma_{f_t})$ as $t \to 0$ by means of Proposition 2, we first have to examine the t-dependence of the amplitude $a^{\iota\tilde{\iota}}_{f_t}(\varphi_\iota(p), \xi; k_1, k)$ as $t \to 0$ for fixed $k, k_1 \in K$. Let $0 \leq \beta \leq 1$ be a test function on G with support in a sufficiently small neighborhood $U = U^{-1}$ of the identity that is identically 1 on a ball of radius $R > 0$ around e, and consider for $f \in \mathcal{S}(G)$

$$\begin{aligned} {}^1 a^{\iota\tilde{\iota}}_f(x, \xi; k_1, k_2) &= e^{-i\varphi_{\tilde{\iota}}^{k_1 k_2}(x)\cdot\xi} \alpha'_{\tilde{\iota}}(k_1 k_2 \cdot \varphi_\iota^{-1}(x)) \int_G e^{i\varphi_\iota^{k_1 g k_2}(x)\cdot\xi} c_\iota(x, k_1 g k_2) f(g) \\ &\quad \cdot (1 - \beta)(g)\, dg, \\ {}^2 a^{\iota\tilde{\iota}}_f(x, \xi; k_1, k_2) &= e^{-i\varphi_{\tilde{\iota}}^{k_1 k_2}(x)\cdot\xi} \alpha'_{\tilde{\iota}}(k_1 k_2 \cdot \varphi_\iota^{-1}(x)) \int_G e^{i\varphi_\iota^{k_1 g k_2}(x)\cdot\xi} c_\iota(x, k_1 g k_2) f(g) \\ &\quad \cdot \beta(g)\, dg. \end{aligned}$$

Similarly to (11), one has for arbitrary $N \in \mathbb{N}$ the equality

$$\psi^\iota_{\xi,x}(k_1 g k_2)(1 + |\xi|^2)^N = \sum_{r=0}^{2N} \sum_{|\alpha|=r} b^N_\alpha(x, g, k_1, k_2) dL(X^\alpha)[\psi^\iota_{\xi,x}(k_1 g k_2)], \tag{20}$$

where the coefficients $b^N_\alpha(x, g, k_1, k_2)$ are at most of exponential growth in g. With (12) and (19) we obtain

$$|\, {}^1 a^{\iota\tilde{\iota}}_{f_t}(\varphi_\iota(p), \xi; k_1, k)| \leq c(1 + |\xi|^2)^{-N} e^{\omega t} t^{-(d+2N)/q} e^{-bR^{q/(q-1)}[t^{1-q}-1]} \int_G e^{-b|g|^{q/(q-1)}} e^{\kappa|g|}\, dg$$

for small $t > 0$, and constants $b, c > 0$, $\kappa, \omega \geq 0$. Consequently, ${}^1 a^{\iota\tilde{\iota}}_{f_t}(x, \xi; k_1, k)$ vanishes to all orders as $t \to 0$, or $|\xi| \to \infty$, provided that $q \geq 2$, and with Proposition 2 we obtain the equality

$$\begin{aligned} \operatorname{tr} \pi(H^\sigma_{f_t}) &= \operatorname{tr} \pi(H^\sigma_{f_t \beta}) + O(t^N) \\ &= \frac{d_\sigma^2}{(2\pi)^n} \sum_{\iota, \tilde{\iota}} \int_K \int_K \int_{T^*M} e^{i\Phi_{\iota\tilde{\iota}}(p, \xi, k_1, k)} \alpha_\iota(p) \alpha_{\tilde{\iota}}(k_1 k \cdot p) \overline{\chi_\sigma(k_1) \chi_\sigma(k)} \\ &\quad \cdot {}^2 a^{\iota\tilde{\iota}}_{f_t}(\varphi_\iota(p), \xi; k_1, k) j_\iota(p)\, d(T^*M)(p, \xi)\, dk\, dk_1 + O(t^N) \end{aligned}$$

for any $N \in \mathbb{N}$. Let $\psi \in \mathrm{C}^\infty_c(\mathbb{R}^n, \mathbb{R}^+)$ be equal 1 near the origin, and $\varepsilon > 0$. Repeating the arguments in the proof of Proposition 2 with f replaced by $f_t \cdot \beta$ one obtains for $\operatorname{tr} \pi(H^\sigma_{f_t \beta})$ the expression

$$\begin{aligned} &\lim_{\varepsilon \to 0} d_\sigma^2 \sum_\iota \int_{\widetilde{W}_\iota} \int \int_U \int_K \int_K e^{i(\varphi_\iota(k_1 g k \cdot p) - \varphi_\iota(p))\cdot\xi} f_t(g) \beta(g) \alpha_\iota(p) c_\iota(\varphi_\iota(p), k_1 g k) \overline{\chi_\sigma(k_1) \chi_\sigma(k)} \psi(\varepsilon\xi) \\ &\qquad \cdot j_\iota(p)\, dk\, dk_1\, dg\, đ\xi\, dM(p) \\ &= \lim_{\varepsilon \to 0} d_\sigma^2 \sum_\iota \int_U \int_K \int_K \int_{\widetilde{W}_\iota} \int e^{i(\varphi_\iota(k_1 g k \cdot p) - \varphi_\iota(k_1 k \cdot p))\cdot\xi} e^{i(\varphi_\iota(k_1 k \cdot p) - \varphi_\iota(p))\cdot\xi} f_t(g) \beta(g) \alpha_\iota(p) \\ &\qquad \cdot \alpha'_\iota(k_1 g k \cdot p) \overline{\chi_\sigma(k_1) \chi_\sigma(k)} \psi(\varepsilon\xi) j_\iota(p)\, đ\xi\, dM(p)\, dk\, dk_1\, dg. \end{aligned}$$

Here we took into account that $U \subset G$ can be chosen so small that for all $k_1, k_2 \in K$, $g \in U$, and $\iota \in I$

$$k_1 g k_2 \cdot p \in \operatorname{supp} \alpha'_\iota \quad \Longrightarrow \quad k_1 k_2 \cdot p \in \widetilde{W}'_\iota,$$

since I can be assumed to be finite due to the compactness of M. Consequently

$$b^\iota_f(x,\xi;k_1,k_2)=e^{-i\varphi_\iota^{k_1k_2}(x)\cdot\xi}\int_U e^{i\varphi_\iota^{k_1gk_2}(x)\cdot\xi}c_\iota(x,k_1gk_2)f(g)\beta(g)\,dg,\qquad f\in\mathcal{S}(G),$$

is well defined, and $b^\iota_{f_t}(\varphi_\iota(p),\xi;k_1,k)=0$ for $k_1k\cdot p\notin\widetilde{W}'_\iota$. From the considerations in the proof of Theorem 1, and (20) it follows that $b^\iota_f(x,\xi;k_1,k_2)\in S^{-\infty}(W_\iota\times\mathbb{R}^n)$ for arbitrary $k_1,k_2\in K$. Thus, we arrive at

$$\operatorname{tr}\pi(H^\sigma_{f_t\beta})=\lim_{\varepsilon\to0}d^2_\sigma\sum_\iota\int_K\int_K\int_{\widetilde{W}_\iota}\int e^{i(\varphi_\iota(k_1k\cdot p)-\varphi_\iota(p))\cdot\xi}\alpha_\iota(p)\overline{\chi_\sigma(k_1)\chi_\sigma(k)}\psi(\varepsilon\xi)$$
$$\cdot\, b^\iota_{f_t}(\varphi_\iota(p),\xi;k_1,k)j_\iota(p)\,đ\xi\,dM(p)\,dk\,dk_1.$$

By passing to the limit under the integral, and performing the substitution $\xi\to\xi/t^{1/q}$, one finally arrives at the desired result. To examine the t-dependence of the amplitude $b^\iota_{f_t}(\varphi_\iota(p),\xi/t^{1/q};k_1,k)$ as $t\to0$, introduce canonical coordinates on U according to

$$\Psi:\mathbb{R}^d\ni\zeta=(\zeta_1,\dots,\zeta_d)\longmapsto g=e^{\sum\zeta_iX_i}\in U. \tag{21}$$

By the analyticity of the G-action on M we have the power expansion

$$[\varphi_\iota^{k_1k}(x)-\varphi_\iota^{k_1gk}(x)]_j=\sum_{|\alpha|>0}c^j_\alpha(x,k_1,k)\zeta^\alpha,\qquad g\in U,x\in W_\iota,$$

where the coefficients $c^j_\alpha(x,k_1,k)$ depend analytically on x, k_1, and k. Performing the substitution $\zeta\mapsto t^{1/q}\zeta$, and taking into account the bounds (19), one computes

$$|b^\iota_{f_t}(x,\xi/t^{1/q};k_1,k)|=t^{d/q}\Big|\int_{t^{-1/q}\Psi^{-1}(U)}e^{i\sum_{|\alpha|>0,j}c^j_\alpha(x,k_1,k)(t^{1/q}\zeta)^\alpha\xi_j/t^{1/q}}c_\iota(x,k_1e^{t^{1/q}\sum\zeta_iX_i}k)$$
$$\cdot\,(f_t\beta)(e^{t^{1/q}\sum\zeta_iX_i})\Psi^*(d_G)(\zeta)\Big|\leq c'e^{\omega t}\int_{\mathbb{R}^d}e^{-b'|\zeta|^{q/(q-1)}}\Psi^*(d_G)(\zeta)$$

for some constants $b',c'>0$, and $\omega\geq0$, where we took into account that there exists some constant $C>0$ such that $C^{-1}|\zeta|\leq|g|\leq C|\zeta|$. A similar examination of the derivatives finally yields for small $t>0$ the estimate

$$|\,\partial^\alpha_x\,\partial^\beta_\xi\,\partial^{\delta_1}_{k_1}\,\partial^\delta_k[b^\iota_{f_t}(x,\xi/t^{1/q};k_1,k)]|\leq C$$

for some constant $C>0$ independent of $0<t<1$, and arbitrary indices $\alpha,\beta,\delta_1,\delta$. □

Remark 1. Note that since $b^\iota_{f_t}$ is rapidly decaying in ξ, for any $N\in\mathbb{N}$ there exists a constant $c_N>0$ such that

$$|b^\iota_{f_t}(\varphi_\iota(p),\xi/t^{1/q};k_1,k)|\leq\frac{c_N}{(1+|\xi/t^{1/q}|^2)^N}=\frac{c_Nt^{2N/q}}{(t^{2/q}+|\xi|^2)^N}\leq\frac{c_Nt^{2N/q}}{|\xi|^{2N}}.$$

Therefore, if $\theta\in\mathrm{C^\infty_c}(\mathbb{R}^n,[0,1])$ is a cut-off function such that $\theta(\xi)=1$ for $|\xi|\leq1$, and $\theta(\xi)=0$ for $|\xi|\geq2$, then

$$\int_K\int_K\int_{T^*M}|b^\iota_{f_t}(\varphi_\iota(p),\xi/t^{1/q};k_1,k)|(1-\theta(\xi))d(T^*M)(p,\xi)\,dk\,dk_1\leq C_Nt^{2N/q}$$

for any $N\in\mathbb{N}$, and suitable constants C_N.

Let us now regard the compact group

$$\mathbb{K}=K\times K$$

with Haar measure $d_{\mathbb{K}} = d_K d_K$. Take $\sigma \in \widehat{K}$, and $(\pi_\sigma, V_\sigma) \in \sigma$. Then $(\pi_\sigma \otimes \pi_\sigma, V_\sigma \otimes V_\sigma)$ is an unitary irreducible representation of $\mathbb{K}$ belonging to $\sigma \otimes \sigma \in \hat{\mathbb{K}}$ of dimension $d_{\sigma\otimes\sigma} = d_\sigma^2$, and the corresponding character is given by

$$(\chi_\sigma \otimes \chi_\sigma)(k_1, k) = \chi_\sigma(k_1) \otimes \chi_\sigma(k) = \chi_\sigma(k_1)\chi_\sigma(k), \qquad k_1, k \in K.$$

In what follows, we shall also write $(k_1, k) \cdot p = k_1 k \cdot p$ for the $\mathbb{K}$-action on M. Note that this action is still isometric, but no longer effective.

Corollary 3. *Let $\sigma \in \widehat{K}$, and $\psi \in \mathrm{C_c^\infty}(\mathbb{R}^n, \mathbb{R}^+)$ be equal 1 near the origin. Let further $t, \varepsilon > 0$. Then*

$$\operatorname{tr} \pi(H^\sigma_{f_t}) = \lim_{\varepsilon \to 0} \frac{d_{\sigma\otimes\sigma}}{(2\pi)^n t^{n/q}} \sum_\iota \int_{\mathbb{K}} \int_{T^*M} e^{i\Phi_{\iota\iota}(p,\xi,k_1,k)/t^{1/q}} \alpha_\iota(p) \overline{(\chi_\sigma \otimes \chi_\sigma)(k_1, k)}$$
$$\cdot\, b^\iota_{f_t}(\varphi_\iota(p), \xi/t^{1/q}; k_1, k) \psi(\varepsilon\xi) j_\iota(p)\, d(T^*M)(p, \xi) d_{\mathbb{K}}(k_1, k) + O(t^\infty).$$

Proof. This is an immediate consequence of Theorem 2, and Lebesgue's theorem on bounded convergence. □

4. Singular equivariant asymptotics and resolution of singularities

The considerations of the previous section showed that, in order to describe the traces $\operatorname{tr} \pi(H^\sigma_{f_t}) = \operatorname{tr}(P_\sigma \circ \pi(f_t) \circ P_\sigma)$ as $t \to 0^+$, one has to study the asymptotic behavior of oscillatory integrals of the form

$$I(\mu) = \int_{\mathbb{K}} \int_{T^*\widetilde{W}} e^{i\Phi(p,\xi,k_1,k)/\mu} a(k_1 k \cdot p, p, \xi, k_1, k)\, d(T^*M)(p, \xi) d_{\mathbb{K}}(k_1, k) \tag{22}$$

as $\mu \to 0^+$ by means of the stationary phase principle, where $(\varphi, \widetilde{W})$ are local coordinates on M, while $a \in \mathrm{C_c^\infty}(\widetilde{W} \times T^*\widetilde{W} \times \mathbb{K})$ is an amplitude which might depend on μ, and

$$\Phi(p, \xi, k_1, k) = (\varphi((k_1, k) \cdot p) - \varphi(p)) \cdot \xi. \tag{23}$$

Consider for this the cotangent bundle $\pi : T^*M \to M$, as well as the tangent bundle $\tau : T(T^*M) \to T^*M$, and define on T^*M the Liouville form

$$\Theta(\mathfrak{X}) = \tau(\mathfrak{X})[\pi_*(\mathfrak{X})], \qquad \mathfrak{X} \in T(T^*M).$$

Regard T^*M as a symplectic manifold with symplectic form $\omega = d\Theta$, and define for any element X in the Lie algebra $\mathfrak{k} \oplus \mathfrak{k}$ of $\mathbb{K}$ the function

$$J_X : T^*M \longrightarrow \mathbb{R}, \quad \eta \mapsto \Theta(\widetilde{X})(\eta),$$

where $\widetilde{X}$ denotes the fundamental vector field on T^*M, respectively M, generated by X. $\mathbb{K}$ acts on T^*M in a Hamiltonian way, and the corresponding symplectic momentum map is given by

$$\mathbb{J} : T^*M \to (\mathfrak{k} \oplus \mathfrak{k})^*, \quad \mathbb{J}(\eta)(X) = J_X(\eta).$$

Let us next compute the critical set of the phase function Φ. Clearly, $\partial_\xi \Phi(p, \xi, k_1, k) = 0$ if, and only if $k_1 k \cdot p = p$. Write $\varphi(p) = (x_1, \dots, x_n)$, $\eta = \sum \xi_i (dx_i)_p \in T^*_p\widetilde{W}$. Assuming that $k_1 k \cdot p = p$, one computes for any $X \in \mathfrak{k} \oplus \mathfrak{k}$

$$\frac{d}{ds}\Big(\varphi(\mathrm{e}^{-tX}(k_1, k) \cdot p) \cdot \xi\Big)_{|t=0} = \sum \xi_i \widetilde{X}_p(x_i) = \sum \xi_i (dx_i)_p(\widetilde{X}_p) = \eta(\widetilde{X}_p) = \Theta(\widetilde{X})(\eta) = \mathbb{J}(\eta)(X),$$

so that $\partial_{(k_1,k)} \Phi(p, \xi, k_1, k) = 0$ if, and only if $\mathbb{J}(\eta) = 0$. A further computation shows that

$$\partial_x \Phi(\varphi^{-1}(x), \xi, k_1, k) = [\,^T(\varphi \circ k_1 k \circ \varphi^{-1})_{*,x} - \mathbf{1}]\xi = ((k_1 k)^*_x - \mathbf{1}) \cdot \xi,$$

so that $\partial_p \Phi(p,\xi,k_1,k)=0$ amounts precisely to the condition $(k_1k)^*\xi=\xi$. Collecting everything together one obtains

$$\begin{aligned}\mathrm{Crit}(\Phi) &= \Big\{(p,\xi,k_1,k)\in T^*\widetilde{W}\times\mathbb{K} : (\Phi_*)_{(p,\xi,k_1,k)}=0\Big\}\\ &= \Big\{(p,\xi,k_1,k)\in (\Xi\cap T^*\widetilde{W})\times\mathbb{K} :\ (k_1,k)\cdot(p,\xi)=(p,\xi)\Big\},\end{aligned}\tag{24}$$

where $\Xi=\mathbb{J}^{-1}(0)$ denotes the zero level of the momentum map of $\mathbb{K}$. Now, the major difficulty resides in the fact that, unless the $\mathbb{K}$-action on T^*M is free, the considered momentum map is not a submersion, so that Ξ and $\mathrm{Crit}(\Phi)$ are not smooth manifolds. The stationary phase theorem can therefore not immediately be applied to the integrals $I(\mu)$. Nevertheless, it was shown in [21] that by constructing a strong resolution of the set

$$\mathcal{N}=\{(p,k_1,k)\in M\times\mathbb{K} : (k_1,k)\cdot p=p\}$$

a partial desingularization $\mathcal{Z}:\widetilde{\mathbf{X}}\to\mathbf{X}=T^*M\times\mathbb{K}$ of the set

$$\mathcal{C}=\{(p,\xi,k_1,k)\in\Xi\times\mathbb{K} :\ (k_1,k)\cdot(p,\xi)=(p,\xi)\}$$

can be achieved, and applying the stationary phase theorem in the resolution space, an asymptotic description of $I(\mu)$ can be obtained. More precisely, the map $\mathcal{Z}$ yields a partial monomialization of the local ideal $I_\Phi=(\Phi)$ generated by the phase function (23) according to

$$\mathcal{Z}^*(I_\Phi)\cdot\mathcal{E}_{\tilde{x},\widetilde{\mathbf{X}}}=\prod_j \sigma_j^{l_j}\cdot\mathcal{Z}_*^{-1}(I_\Phi)\cdot\mathcal{E}_{\tilde{x},\widetilde{\mathbf{X}}},$$

where $\mathcal{E}_{\widetilde{\mathbf{X}}}$ denotes the structure sheaf of rings of $\widetilde{\mathbf{X}}$, σ_j are local coordinate functions near each $\tilde{x}\in\widetilde{\mathbf{X}}$, and l_j natural numbers. As a consequence, the phase function factorizes locally according to $\Phi\circ\mathcal{Z}\equiv\prod\sigma_j^{l_j}\cdot\tilde{\Phi}^{wk}$, and one shows that the weak transforms $\tilde{\Phi}^{wk}$ have clean critical sets. Asymptotics for the integrals $I(\mu)$ are then obtained by pulling them back to the resolution space $\widetilde{\mathbf{X}}$, and applying the stationary phase theorem to the $\tilde{\Phi}^{wk}$ with the variables σ_j as parameters. As a consequence, one obtains

Theorem 3. *Let M be a connected, closed Riemannian manifold, and K a compact, connected Lie group acting isometrically on M. For $\mathbb{K}=K\times K$, consider the oscillatory integral*

$$I(\mu)=\int_{\mathbb{K}}\int_{T^*\widetilde{W}} e^{i\Phi(p,\xi,k_1,k)/\mu}a((k_1,k)\cdot p,p,\xi,k_1,k)\,d(T^*M)(p,\xi)d_{\mathbb{K}}(k_1,k),\qquad \mu>0,$$

where $(\varphi,\widetilde{W})$ are local coordinates on M, while $a\in \mathrm{C}^\infty_{\mathrm{c}}(\widetilde{W}\times T^*\widetilde{W}\times\mathbb{K})$ is an amplitude which might depend on the parameter μ, and $\Phi(p,\xi,k_1,k)=(\varphi((k_1,k)\cdot p)-\varphi(p))\cdot\xi$. Furthermore, assume that for all multi-indices one has $|\,\partial_x^\alpha\,\partial_\xi^\beta\,\partial_{k_1}^{\delta_1}\partial_k^\delta\,a|\leq C$ with a constant $C>0$ independent of μ. Then $I(\mu)$ has the asymptotic expansion

$$I(\mu)=(2\pi\mu)^\kappa\mathcal{L}_0+O\big(\mu^{\kappa+1}(\log\mu^{-1})^{\Lambda-1}\big),\qquad \mu\to0^+,$$

where κ is the dimension of a $\mathbb{K}$-orbit of principal type in M, Λ the maximal number of elements of a totally ordered subset of the set of $\mathbb{K}$-isotropy types, and the leading coefficient is given by

$$\mathcal{L}_0=\int_{\mathrm{Reg}\,\mathcal{C}}\frac{a(k_1k\cdot p,p,\xi,k_1,k)}{|\det\,\Phi''(p,\xi,k_1,k)_{N_{(p,\xi,k_1,k)}\mathrm{Reg}\,\mathcal{C}}|^{1/2}}\,d(\mathrm{Reg}\,\mathcal{C})(p,\xi,k_1,k),\tag{25}$$

where $\mathrm{Reg}\,\mathcal{C}$ denotes the regular part of $\mathcal{C}$, and $d(\mathrm{Reg}\,\mathcal{C})$ the induced volume density. In particular, the integral over $\mathrm{Reg}\,\mathcal{C}$ exists.

Proof. See [21], Theorem 11. □

As a consequence, one obtains the following asymptotic description as $t \to 0$ for $\operatorname{tr}\pi(H^\sigma_{f_t}) = \operatorname{tr}(P_\sigma \circ \pi(f_t) \circ P_\sigma)$.

Theorem 4. *Let $\sigma \in \widehat{K}$, and $t > 0$. Then*

$$\operatorname{tr}\pi(H^\sigma_{f_t}) = \frac{d_{\sigma\otimes\sigma}}{(2\pi)^{n-\kappa}t^{(n-\kappa)/q}} \sum_\iota \int_{\operatorname{Reg}\mathcal{C}} \alpha_\iota(p) b^\iota_{f_t}(\varphi_\iota(p), \xi/t^{1/q}; k_1, k) \overline{(\chi_\sigma \otimes \chi_\sigma)(k_1, k)} j_\iota(p) \cdot \frac{d(\operatorname{Reg}\mathcal{C})(p,\xi,k_1,k)}{|\det \Phi''_{\iota\iota}(p,\xi,k_1,k)_{N_{(p,\xi,k_1,k)}\operatorname{Reg}\mathcal{C}}|^{1/2}} + O(t^{-(n-\kappa-1)/q}(\log t)^{\Lambda-1}),$$

where κ is the dimension of a K-orbit of principal type in M, and Λ the maximal number of elements of a totally ordered subset of the set of $\mathbb{K}$-isotropy types.

Proof. This is an immediate consequence of Corollary 3, and Theorem 3, together with Lebesgue's theorem on bounded convergence. □

In general, it is not possible to obtain more explicit expressions for the leading term, unless one has more knowledge about the Langlands kernels f_t as $t \to 0$. In particular, the bounds (19) are not sufficient for this purpose. We shall therefore make the following assumption, which should hold in many cases.

Assumption 1. The function f_t has an asymptotic expansion of the form

$$f_t(g) \sim \frac{1}{t^{d/q}} e^{-b\left(\frac{|g|^q}{t}\right)^{1/(q-1)}} \sum_{j=0}^\infty c_j(g) t^j, \qquad |g| << 1,$$

where $b > 0$, and the coefficients $c_j(g)$ are analytic in g.

We then have the following

Corollary 4. *Let Assumption 1 be fulfilled. Then*

$$\operatorname{tr}\pi(H^\sigma_{f_t}) = \frac{d_{\sigma\otimes\sigma}}{(2\pi)^{n-\kappa}t^{(n-\kappa)/q}} [(\pi_\sigma \otimes \pi_\sigma)_{|\mathbb{H}} : \mathbf{1}] \sum_\iota \int_{\operatorname{Reg}\Xi} \hat{\mathcal{F}}_\iota(p,\xi)\alpha_\iota(p) j_\iota(p) \frac{d(\operatorname{Reg}\Xi)(p,\xi)}{vol\,\mathcal{O}_{(p,\xi)}} + O(t^{-(n-\kappa-1)/q}(\log t)^{\Lambda-1}),$$

where $\hat{\mathcal{F}}_\iota(p,\xi) = c_0(e) \int_{\mathbb{R}^d} e^{i\sum_{l,j} c^j_l(p)\zeta_l\xi_j} e^{-b|e^{\sum \zeta_i X_i}|^{q/(q-1)}} \Psi^*(d_G)(\zeta)$ is rapidly falling in ξ, and $\mathcal{O}_{(p,\xi)}$ denotes the $\mathbb{K}$-orbit in T^*M through (p,ξ), while $[(\pi_\sigma \otimes \pi_\sigma)_{|\mathbb{H}} : \mathbf{1}]$ is the multiplicity of the trivial representation in the restriction of the unitary irreducible representation $\pi_\sigma \otimes \pi_\sigma$ to a principal isotopy group $\mathbb{H} \subset \mathbb{K}$. Actually,

$$\widetilde{vol}(\Xi/\mathbb{K}) = \sum_\iota \int_{\operatorname{Reg}\Xi} \hat{\mathcal{F}}_\iota(p,\xi)\alpha_\iota(p) j_\iota(p) \frac{d(\operatorname{Reg}\Xi)(p,\xi)}{vol\,\mathcal{O}_{(p,\xi)}}$$

represents a Gaussian volume of the symplectic quotient $\Xi/\mathbb{K}$.

Proof. On $\operatorname{Reg}\mathcal{C}$ we have $k_1 k \cdot p = p$, so that

$$b^\iota_{f_t}(\varphi_\iota(p), \xi; k_1, k) = e^{-i\varphi_\iota(p)\cdot\xi} \int_U e^{i\varphi_\iota(k_1 g k_1^{-1}\cdot p)\cdot\xi} \alpha'_\iota(k_1 g k_1^{-1}\cdot p) f_t(g)\beta(g)\, dg = \int_U e^{i[\varphi_\iota(g\cdot p) - \varphi_\iota(p)]\cdot\xi} \alpha'_\iota(g\cdot p)(f_t\beta)(k_1^{-1} g k_1)\, dg,$$

since we can assume that U is invariant under conjugation with K. Consider further, with respect to the coordinates (21), the expansion

$$[\varphi_\iota(g\cdot p)-\varphi_\iota(p)]_j=\sum_{|\alpha|>0} c^j_\alpha(p)\zeta^\alpha, \qquad g\in U,\, p\in \widetilde{W}_\iota,$$

where the coefficients $c^j_\alpha(p)$ depend analytically on p. Under Assumption 1, Taylor expansion in $\tau=t^{1/q}$ at $\tau=0$ gives

$$\begin{aligned} b^\iota_{f_t}(\varphi_\iota(p),\xi/t^{1/q};k_1,k) &= \int_{t^{-1/q}\Psi^{-1}(U)} e^{i\sum_{|\alpha|>0,j} c^j_\alpha(p)(t^{1/q}\zeta)^\alpha\xi_j/t^{1/q}}\alpha'_\iota(e^{t^{1/q}\sum\zeta_iX_i}\cdot p)\\ &\cdot t^{d/q}(f_t\beta)(k_1^{-1}e^{t^{1/q}\sum\zeta_iX_i}k_1)\Psi^*(d_G)(\zeta)\\ &= \int_{t^{-1/q}\Psi^{-1}(U)} e^{i\sum_{|\alpha|>0,j} c^j_\alpha(p)(t^{1/q}\zeta)^\alpha\xi_j/t^{1/q}}\alpha'_\iota(e^{t^{1/q}\sum\zeta_iX_i}\cdot p)c_0\Big(e^{t^{1/q}\sum\zeta_i\mathrm{Ad}\,(k_1^{-1})X_i}\Big)\\ &\cdot e^{-b(|e^{t^{1/q}\sum\zeta_iX_i}|^q/t)^{1/(q-1)}}\beta(k_1^{-1}e^{t^{1/q}\sum\zeta_iX_i}k_1)\Psi^*(d_G)(\zeta)+O(t)\\ &= \alpha'_\iota(p)c_0(e)\int_{\mathbb{R}^d} e^{i\sum_{l,j}c^j_l(p)\zeta_l\xi_j}e^{-b|e^{\sum\zeta_iX_i}|^{q/(q-1)}}\Psi^*(d_G)(\zeta)+O(t^{1/q}), \end{aligned} \tag{26}$$

where the notation is the same as in the proof of Theorem 2. Here we took into account that by Proposition 1 we have $|g|=|kgk^{-1}|$ for all $g\in G$ and $k\in K$. Furthermore, $|e^{t^{1/q}\sum\zeta_iX_i}|^q/t=|e^{\sum\zeta_iX_i}|$. Let us now remark that for any smooth, compactly supported function u on $\Xi\cap T^*\widetilde{W}_\iota$, and any $v\in \mathrm{C}^\infty(\mathbb{K})$, one has the formula

$$\int_{\mathrm{Reg}\,\mathcal{C}}\frac{v(k_1,k)u(p,\xi)d(\mathrm{Reg}\,\mathcal{C})(p,\xi,k_1,k)}{|\det\,\Phi''_{\iota\iota}(p,\xi,k_1,k)_{|N_{(p,\xi,k_1,k)}\mathrm{Reg}\,\mathcal{C}}|^{1/2}}=\int_{\mathbb{H}}v(k_1,k)\,dk_1\,dk\cdot\int_{\mathrm{Reg}\,\Xi}u(p,\xi)\frac{d(\mathrm{Reg}\,\Xi)(p,\xi)}{\mathrm{vol}\,\mathcal{O}_{(p,\xi)}}, \tag{27}$$

compare [6], Lemma 7, where $\mathbb{H}$ is a principal $\mathbb{K}$-isotropy group, and $\mathcal{O}_{(p,\xi)}$ the $\mathbb{K}$-orbit in T^*M through (p,ξ). In particular,

$$\int_{\mathbb{H}}\overline{(\chi_\sigma\otimes\chi_\sigma)(k_1,k)}dk_1dk=[(\pi_\sigma\otimes\pi_\sigma)_{|\mathbb{H}}:\mathbf{1}],$$

where $[(\pi_\sigma\otimes\pi_\sigma)_{|\mathbb{H}}:\mathbf{1}]$ denotes the multiplicity of the trivial representation in the restriction to $\mathbb{H}$ of the unitary irreducible representation $\pi_\sigma\otimes\pi_\sigma$. The assertion now follows with Theorem 4. □

To motivate Assumption 1, and to illustrate our results, let us consider the classical heat kernel on G. Thus, consider a Cartan decomposition of $\mathfrak{g}$ as in (1), and let $X_1,\dots X_p$ be an orthonormal basis of $\mathfrak{p}$, and $Y_1,\dots,Y_l$ an orthonormal basis for $\mathfrak{k}$ with respect to $\langle\cdot,\cdot\rangle_\theta$. If Ω and Ω_K denote the Casimir elements of G and K, one has

$$\Omega=\sum_{i=1}^p X_i^2-\sum_{i=1}^l Y_i^2, \qquad \Omega_K=-\sum_{i=1}^l Y_i^2.$$

Let

$$P=-\Omega+2\Omega_K=-\sum_{i=1}^p X_i^2-\sum_{i=1}^l Y_i^2.$$

Then $dR(P)$ is the Beltrami-Laplace operator Δ_G on G with respect to the left invariant metric. $dR(P)$ is a strongly elliptic operator associated to R, and generates a strongly continuous semigroup which coincides with the classical heat semigroup $e^{-t\Delta_G}$, whose kernel p_t is given by the corresponding universal Langlands kernel. In particular,

$$e^{-t\Delta_G}=R(p_t), \tag{28}$$

see [17], Section 3. Let us now recall that on Riemannian manifolds admitting a properly discontinuous group of isometries with compact quotient, a fundamental solution of the heat equation with Gaussian bounds can be constructed explicitly [9]. Furthermore, every real, semisimple Lie group possesses a discrete, torsion-free subgroup with compact quotient [5]. If therefore $H(t,g,h)$ is the fundamental solution of the heat equation $\partial / \partial t + \Delta_G$ on G constructed in this way, the Gaussian bounds imply that it coincides with the Langlands kernel p_t, so that $H(t,g,h) = p_t(g^{-1}h)$. Furthermore, one has an asymptotic expansion of the form

$$H(t,g,h) \sim (4\pi t)^{-d/2} e^{-\frac{d^2(g,h)}{4t}} \sum_{j=0}^{\infty} t^j u_j(g,h),$$

valid in a sufficiently small neighborhood of the diagonal in $G \times G$, see [9], Theorem 3.3. As before, $d(g,h)$ denotes the geodesic distance between two points with respect to the left invariant metric on G, and $u_0(g,g) = 1$. Corollary 4 then implies

Corollary 5. *Let Δ_G be the Laplace-Beltrami operator on G, and $p_t \in \mathcal{S}(G)$ its heat kernel. Then*

$$\operatorname{tr} \pi(H^\sigma_{p_t}) = \frac{d_{\sigma\otimes\sigma}}{(2\pi)^{n-\kappa} t^{(n-\kappa)/2}} [(\pi_\sigma \otimes \pi_\sigma)_{|\mathbb{H}} : \mathbf{1}] \widetilde{vol}\,(\Xi/\mathbb{K}) + O(t^{-(n-\kappa-1)/2} (\log t)^{\Lambda-1}),$$

where

$$\widetilde{vol}\,(\Xi/\mathbb{K}) = \sum_\iota \int_{\operatorname{Reg}\Xi} \hat{\mathcal{F}}_\iota(p,\xi) \alpha_\iota(p) j_\iota(p) \frac{d(\operatorname{Reg}\Xi)(p,\xi)}{vol\,\mathcal{O}_{(p,\xi)}},$$

and $\hat{\mathcal{F}}_\iota(p,\xi) = (4\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i \sum_{l,j} c_l^j(p) \zeta_l \xi_j} e^{-|e^{\sum \zeta_i X_i}|^2/4} \Psi^*(d_G)(\zeta)$.

□

5. Homogeneous vector bundles on compact locally symmetric spaces

In this section, we apply the previous analysis to heat traces of Bochner-Laplace operators on compact, locally symmetric spaces. In the rank one case, this problem was already considered by Miatello [14] and DeGeorge and Wallach [8]. As before, let G denote a connected, real, semisimple Lie group with finite center, and Γ a discrete, uniform subroup of G. Consider $M = \Gamma\backslash G$, and denote by $\pi_\Gamma(g)\varphi(h) = \varphi(hg)$, $g,\ h \in G$, the right regular representation [2] of G in the space $L^2(\Gamma\backslash G)$ of square integrable functions on $\Gamma\backslash G$. Since $\Gamma\backslash G$ is compact, the right regular representation decomposes discretely according to

$$\pi_\Gamma \simeq \bigoplus_{\varrho \in \widehat{G}} m_\varrho \pi_\varrho, \tag{29}$$

where $\widehat{G}$ stands for the set of equivalence classes of irreducible unitary representations of G, $(\pi_\varrho, H_\varrho) \in \varrho$, and $m_\varrho < \infty$ denotes the multiplicity of ϱ in $(\pi_\Gamma, \mathrm{L}^2(\Gamma\backslash G))$. For $f \in C^\infty_c(G)$, the Bochner integral $\pi_\Gamma(f) = \int_G f(g)\pi_\Gamma(g)\, dg$ defines a bounded operator on $L^2(\Gamma\backslash G)$ whose kernel is given by the C^∞ function

$$k_f(g,h) = \sum_{\gamma\in\Gamma} f(g^{-1}\gamma h), \qquad g,h \in G, \tag{30}$$

the series converging uniformly on compacta. The regularity of the kernel implies that $\pi_\Gamma(f)$ is of trace class, and

$$\operatorname{tr} \pi_\Gamma(f) = \int_{\Gamma\backslash G} k_f(g,g)\, dg = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(g^{-1}\gamma g)\, dg.$$

[2] More precisely, $\pi_\Gamma(g)\varphi(\Gamma h) = \varphi(\Gamma h g)$, where $\Gamma h \in \Gamma\backslash G$, $g \in G$.

Note that we are slightly abusing of notation, and denoting the invariant measure on $\Gamma\backslash G$ also by dg. If $f \in L^1(G)$, the operator $\pi_\Gamma(f)$ is still defined, but might not be of trace class. If $f \in \mathcal{S}(G)$ is rapidly falling, it was shown in Theorem 1 that $\pi_\Gamma(f)$ is a smooth operator, which by Corollary 1 implies that it has a well-defined trace. As in the case of a compactly supported f, one can show that for $f \in \mathcal{S}(G)$ the kernel of $\pi_\Gamma(f)$ is given globally by the expression (30), and that it satisfies Selberg's trace formula. Indeed, one has the following

Lemma 2. *Let $f \in \mathcal{S}(G)$ be a rapidly decaying function on G. Then the series $k_f(g,h) = \sum_{\gamma\in\Gamma} f(h^{-1}\gamma g)$ converges uniformly on compacta to a C^∞ function, and represents the integral kernel of the bounded operator $\pi_\Gamma(f) : \mathrm{L}^2(\Gamma\backslash G) \to \mathrm{L}^2(\Gamma\backslash G)$.*

Proof. By Definition 1, for all $\kappa > 0$ we have the inequality

$$|f(h^{-1}\gamma g)| \leq C_\kappa e^{-\kappa|h^{-1}\gamma g|}, \qquad g, h \in G,$$

as well as for all derivatives of all orders of f. Consequently,

$$\sum_{\gamma\in\Gamma} |f(h^{-1}\gamma g)| \leq \sum_{\gamma\in\Gamma} e^{-\kappa\, d(h^{-1}\gamma g, e)} = \sum_{\gamma\in\Gamma} e^{-\kappa\, d(\gamma g, h)}, \tag{31}$$

since left-translation by h is an isometry. Now, recall that for a metric space $(\mathbf{X}, d)$, and a discrete infinite subgroup $\Gamma' \subset \mathrm{Iso}(\mathbf{X})$ of the isometry group of $\mathbf{X}$ the corresponding Poincaré series is defined by

$$P(s,p,q) = \sum_{\gamma\in\Gamma'} e^{-s\, d(p,\gamma q)}, \qquad p, q \in \mathbf{X}, \quad s > 0. \tag{32}$$

By general theory [18], for each discrete subgroup Γ', there exists a $\delta_{\Gamma'} > 0$, called the critical exponent of Γ', such that $P(s,p,q)$ converges for $s > \delta_{\Gamma'}$ and diverges for $s < \delta_{\Gamma'}$. Furthermore, the exponent $\delta_{\Gamma'}$ does not depend on p or q. The estimate (31) means that for fixed $g, h \in G$, the series $k_f(g,h)$ is majorized by the Poincaré series $\sum_{\gamma\in\Gamma} e^{-\kappa\, d(\gamma g, h)}$. Choosing $\kappa > \delta_\Gamma$, we deduce that $k_f(g,h)$ is absolutely convergent for fixed $g, h \in G$. To see that $(g,h) \mapsto k_f(g,h)$ is continuous, note that

$$\begin{aligned}\Big|k_f(h,g) - k_f(z,g)\Big| &= \Big|\sum_{\gamma\in\Gamma} f(h^{-1}\gamma g) - \sum_{\gamma\in\Gamma} f(z^{-1}\gamma g)\Big| \\ &\leq \sum_{\substack{\gamma\in\Gamma \\ \|\gamma\|\leq R}} \Big|f(h^{-1}\gamma g) - f(z^{-1}\gamma g)\Big| + \Big|\sum_{\substack{\gamma\in\Gamma \\ \|\gamma\|>R}} f(h^{-1}\gamma g)\Big| + \Big|\sum_{\substack{\gamma\in\Gamma \\ \|\gamma\|>R}} f(z^{-1}\gamma g)\Big|.\end{aligned}$$

Since the sum $\sum_{\gamma\in\Gamma} f(h^{-1}\gamma g)$ converges, the last two terms in the last inequality can be made as small as required by choosing R big enough, while the first term becomes small if $d(h,z)$ is small, being a finite some of continuous functions. Thus, $(g,h) \mapsto k_f(g,h)$ is continuous. Since the same argument works for all derivatives, $f(h,g)$ converges uniformly on compacta to a C^∞ function. To see that $k_f(g,h)$ represents the Schwartz kernel of $\pi_\Gamma(f)$ for $f \in \mathcal{S}(G)$, note that $\pi_\Gamma(f)$ acts on $\varphi \in L^2(\Gamma\backslash G)$ according to

$$(\pi_\Gamma(f)\varphi)(h) = \int_G f(g)\varphi(hg)\, dg = \int_G f(h^{-1}g)\varphi(g)\, dg, \qquad h \in G, \tag{33}$$

the integral being absolutely convergent due to the inequality

$$|\alpha\beta| \leq \frac{1}{2}|\alpha|^2 + \frac{1}{2}|\beta|^2, \quad \alpha,\, \beta \in \mathbb{C},$$

and the fact that if f is rapidly decreasing, $f \cdot \overline{f}$ is rapidly decreasing, too. By Fubini's theorem, and the first part of the lemma we therefore obtain for each $h \in G$

$$(\pi_\Gamma(f)\varphi)(h) = \int_{\Gamma\backslash G} \left(\sum_{\gamma\in\Gamma} f(h^{-1}\gamma g)\varphi(\gamma g) \right) dg = \int_{\Gamma\backslash G} k_f(h,g)\varphi(g)\, dg,$$

since $\varphi(\gamma g) = \varphi(g)$. Thus, $\pi_\Gamma(f)$ is an integral operator with kernel $k_f(h,g) \in \mathrm{C}^\infty(\Gamma\backslash G \times \Gamma\backslash G)$. □

Corollary 6. *Let $f \in \mathcal{S}(G)$. Then f satisfies Selberg's trace formula*

$$\bigoplus_{\varrho\in\widehat{G}} m_\varrho \operatorname{tr} \pi_\varrho(f) = \sum_{[\gamma]} vol\,(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\, dg, \tag{34}$$

where $[\gamma]$ denotes the conjugacy class of γ in Γ, and Γ_γ and G_γ are the centralizers of γ in Γ and G, respectively.

Proof. Lemma 1 and 2 yield

$$\operatorname{tr} \pi_\Gamma(f) = \int_{\Gamma\backslash G} k_f(g,g)dx = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(g^{-1}\gamma g)\, dg.$$

Denoting by $[\gamma]$ the conjugacy class of γ, and by Γ_γ the centralizer of γ in Γ one deduces

$$\operatorname{tr} \pi_\Gamma(f) = \int_{\Gamma\backslash G} \sum_{[\gamma]} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(g^{-1}\delta^{-1}\gamma\delta g)\, dg = \sum_{[\gamma]} \int_{\Gamma\backslash G} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(g^{-1}\delta^{-1}\gamma\delta g)\, dg,$$

everything being uniformly convergent. Replacing the inner sum by an integral with a counting measure $d\delta$ yields

$$\operatorname{tr} \pi_\Gamma(f) = \sum_{[\gamma]} \int_{\Gamma\backslash G} \int_{\Gamma_\gamma\backslash\Gamma} f(g^{-1}\delta^{-1}\gamma\delta g) d\delta\, dg = \sum_{[\gamma]} \int_{\Gamma_\gamma\backslash G} f(y^{-1}\gamma y) dy,$$

where we took into account that for any sequence $G_1 \subset G_2 \subset G$ of unimodular groups, a right invariant measure on $G_1\backslash G$ can be written as the product of right invariant measures on $G_2\backslash G$, and $G_1\backslash G_2$, respectively. With the same argument, the above equality can be rewritten as

$$\operatorname{tr} \pi_\Gamma(f) = \sum_{[\gamma]} \int_{G_\gamma\backslash G} \int_{\Gamma_\gamma\backslash G_\gamma} f(v^{-1}u^{-1}\gamma uv) du dv,$$

where G_γ denotes the centralizer of γ in G. Since $u^{-1}\gamma u = \gamma$, and $\Gamma_\gamma\backslash G_\gamma$ is compact, one finally obtains the geometric side of the trace formula

$$\operatorname{tr} \pi_\Gamma(f) = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\, dg.$$

To obtain the spectral side, note that according to the decomposition (29) we have

$$\operatorname{tr} \pi_\Gamma(f) = \bigoplus_{\varrho\in\widehat{G}} m_\varrho \operatorname{tr} \pi_\varrho(f),$$

where $\pi_\varrho(f) = \int_G f(g)\pi_\varrho(g)\, dg$ is of trace class, and defines a distribution

$$\theta_\varrho : \mathrm{C}_c^\infty(G) \ni f \mapsto \operatorname{tr} \pi_\varrho(f) \in \mathbb{C}$$

on G which represents the global character of ϱ. Selberg's trace formula for $f \in \mathcal{S}(G)$ now follows. □

Consider next a maximal compact subgroup K of G, and $\sigma \in \widehat{K}$. As a consequence of Theorem 4, and Selberg's formula (34) we obtain

Proposition 3. *Let $f_t \in \mathcal{S}(G)$, $t > 0$, be the Langlands kernel of a semigroup generated by a strongly elliptic operator associated to the representation π_Γ. Then*

$$(L_\sigma f_t)(e) = \frac{d_{\sigma\otimes\sigma}}{(2\pi)^{\dim G/K} vol\,(\Gamma\backslash G)\, t^{\frac{\dim G/K}{q}}} \sum_\iota \int_{\mathrm{Reg}\,\mathcal{C}} \alpha_\iota(p) b^\iota_{f_t}(\varphi_\iota(p), \xi/t^{1/q}; k_1, k)\overline{(\chi_\sigma \otimes \chi_\sigma)(k_1, k)}$$
$$\cdot j_\iota(p) \frac{d(\mathrm{Reg}\,\mathcal{C})(p,\xi,k_1,k)}{|\det\, \Phi''_{\iota\iota}(p,\xi,k_1,k)_{N_{(p,\xi,k_1,k)}\mathrm{Reg}\,\mathcal{C}}|^{1/2}},$$

up to terms of order $O(t^{-(\dim G/K-1)/q}(\log t)^{\Lambda-1})$, the notation being as in Theorem 4. Here L_σ denotes the projector onto the isotypic component $\mathrm{L}^2(G)_\sigma$. If, in addition, Assumption 1 is satisfied, the leading term of $(L_\sigma f_t)(e)$ is given by

$$\frac{d_{\sigma\otimes\sigma}[(\pi_\sigma \otimes \pi_\sigma)_{|\mathbb{H}} : \mathbf{1}]}{(2\pi)^{\dim G/K} vol\,(\Gamma\backslash G)\, t^{\frac{\dim G/K}{q}}} \widetilde{vol}\,(\Xi/\mathbb{K}),$$

where

$$\widetilde{vol}\,(\Xi/\mathbb{K}) = \sum_\iota \int_{\mathrm{Reg}\,\Xi} \hat{\mathcal{F}}_\iota(p,\xi)\alpha_\iota(p) j_\iota(p) \frac{d(\mathrm{Reg}\,\Xi)(p,\xi)}{vol\,\mathcal{O}_{(p,\xi)}},$$

and $\hat{\mathcal{F}}_\iota(p,\xi) = c_0(e)\int_{\mathbb{R}^d} e^{i\sum_{l,j} c^j_l(p)\zeta_l\xi_j} e^{-b|e^{\sum \zeta_i X_i}|^{q/(q-1)}}\Psi^*(d_G)(\zeta)$.

Proof. By Theorem 2, $\mathrm{tr}\,\pi_\Gamma(H^\sigma_{f_t\beta}) = \mathrm{tr}\,\pi_\Gamma(H^\sigma_{f_t}) + O(t^\infty)$, where $0 \le \beta \le 1$ is a test function on G with support in a sufficiently small neighborhood U of $e \in G$, and which is equal 1 close to e. Furthermore, by Theorem 4,

$$\mathrm{tr}\,\pi_\Gamma(H^\sigma_{f_t}) = \frac{d_{\sigma\otimes\sigma}}{(2\pi)^{\dim G/K} t^{\frac{\dim G/K}{q}}} \sum_\iota \int_{\mathrm{Reg}\,\mathcal{C}} \alpha_\iota(p) b^\iota_{f_t}(\varphi_\iota(p), \xi/t^{1/q}; k_1, k)\overline{(\chi_\sigma \otimes \chi_\sigma)(k_1, k)} j_\iota(p)$$
$$\cdot \frac{d(\mathrm{Reg}\,\mathcal{C})(p,\xi,k_1,k)}{|\det\, \Phi''_{\iota\iota}(p,\xi,k_1,k)_{N_{(p,\xi,k_1,k)}\mathrm{Reg}\,\mathcal{C}}|^{1/2}} + O(t^{-(\dim G/K-1)/q}(\log t)^{\Lambda-1}).$$

Next, recall that for any $\gamma \in \Gamma \subset G$, the G-conjugacy class $[\gamma]_G$ is closed. Furthermore, every compactum in G meets only finitely many $[\gamma]_G$, see [16], Lemma 8.1. Consequently, by choosing the support of β sufficiently small, we obtain with (34)

$$\mathrm{tr}\,\pi_\Gamma(H^\sigma_{f_t\beta}) = \mathrm{vol}\,(\Gamma\backslash G)\, H^\sigma_{f_t}(e),$$

and the assertion follows with Theorem 4, and Corollary 4. □

We now apply our results to heat kernels of Bochner-Laplace operators on compact, locally symmetric spaces. Let (π_σ, V_σ) be an irreducible unitary representation of K of class $\sigma \in \widehat{K}$. Consider the associated homogeneous vector bundle $\widetilde{E}_\sigma = (G \times V_\sigma)/K$ over G/K, and endow it with the G-invariant Hermitian fibre metric induced by the inner product in V_σ. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ as in (1), and consider the unique G-invariant connection $\widetilde{\nabla}$ on $\widetilde{E}_\sigma$ given by the condition that if s is a smooth cross section, $Y \in \mathfrak{p}$, and $\Pi : G \to G/K$ is the canonical projection, then

$$\widetilde{\nabla}_{\Pi_*(Y)}(s) = \frac{d}{dt} s(\mathrm{e}^{tY}\, K)_{|s=0},$$

Π_* being the differential of Π at $e \in G$. Let further $\widetilde{\Delta}_\sigma = \widetilde{\nabla}^*\widetilde{\nabla}$ be the Bochner Laplace operator of $\widetilde{\nabla}$, and denote by $\mathrm{C}^\infty(\widetilde{E}_\sigma)$, $\mathrm{C}^\infty_{\mathrm{c}}(\widetilde{E}_\sigma)$, and $\mathrm{L}^2(\widetilde{E}_\sigma)$ the usual spaces of sections of $\widetilde{E}_\sigma$. With respect to the identification

$$\mathrm{C}^\infty(\widetilde{E}_\sigma) = (\mathrm{C}^\infty(G) \otimes V_\sigma)^K,$$

where $(\mathrm{C}^\infty(G)\otimes V_\sigma)^K = \{\varphi: G\to V_\sigma \text{ is smooth and } \varphi(gk)=\pi_\sigma(k)^{-1}\varphi(g),\, k\in K,\, g\in G\}$, and the corresponding identifications for $\mathrm{C}^\infty_c(\widetilde{E}_\sigma)$ and $\mathrm{L}^2(\widetilde{E}_\sigma)$, one has

$$\widetilde{\Delta}_\sigma = -dR(\Omega)\otimes \mathrm{id} + \mathrm{id}\otimes d\pi_\sigma(\Omega_K) = -dR(\Omega)\otimes\mathrm{id} + \lambda_\sigma \mathrm{id}$$

for some $\lambda_\sigma \geq 0$, Ω and Ω_K being the Casimir elements of G and K, respectively, see [14], Proposition 1.1. As it turns out, the operator $\widetilde{\Delta}_\sigma:\mathrm{C}^\infty_c(\widetilde{E}_\sigma)\to \mathrm{L}^2(\widetilde{E}_\sigma)$ is essentially self-adjoint, and has a unique self-adjoint extension which we shall also denote by $\widetilde{\Delta}_\sigma$. It is a positive operator, and we denote the corresponding heat semigroup by $e^{-t\widetilde{\Delta}_\sigma}$. It is given by

$$(e^{-t\widetilde{\Delta}_\sigma}\varphi)(g) = \int_G h^\sigma_t(g_1)\varphi(gg_1)\,dg_1, \quad \varphi\in(\mathrm{L}^2(G)\otimes V_\sigma)^K, \tag{35}$$

where $h^\sigma_t: G\to \mathrm{End}(V_\sigma)$ is square integrable, and has the covariance property

$$h^\sigma_t(g) = \pi_\sigma(k)h^\sigma_t(k^{-1}gk_1)\pi_\sigma(k_1)^{-1}, \qquad g\in G,\, k,k_1\in K.$$

As one can show, h^σ_t is actually given in terms of the classical heat kernel p_t introduced in (28) according to

$$h^\sigma_t(g) = e^{t\lambda_\sigma}\int_K\int_K p_t(k^{-1}gk_1)\pi_\sigma(kk_1^{-1})\,dk_1\,dk, \tag{36}$$

see [2] and [17], Section 3. Let now Γ be a discrete, uniform, torsion-free subgroup of G. Then Γ acts without fixed points on G/K, and $\Gamma\backslash G/K$ constitutes a compact, locally symmetric space. Let $E_\sigma = \Gamma\backslash\widetilde{E}_\sigma\to\Gamma\backslash G/K$ be the pushdown of the homogenous vector bundle $\widetilde{E}_\sigma\to G/K$. Again, we have identification

$$\mathrm{C}^\infty(E_\sigma) = (\mathrm{C}^\infty(\Gamma\backslash G)\otimes V_\sigma)^K,$$

and similarly for $\mathrm{C}^\infty_c(E_\sigma)$, and $\mathrm{L}^2(E_\sigma)$. Since $\widetilde{\Delta}_\sigma$ is G-invariant, it induces an elliptic, essentially self-adjoint operator $\Delta_\sigma=\nabla^*\nabla:\mathrm{C}^\infty(E_\sigma)\to\mathrm{L}^2(E_\sigma)$, where ∇ is the pushdown of the canonical connection $\widetilde{\nabla}$. Let $e^{-t\Delta_\sigma}$ be the corresponding heat semigroup. With respect to a basis $\{e_i\}$ of V_σ, we obtain with (35) and (36)

$$[e^{-t\Delta_\sigma}\varphi)(g)]_j = \sum_{k=1}^{\dim\sigma}\pi_\Gamma({}^{jk}H^\sigma_t)[\varphi(g)]_k, \quad \varphi\in(\mathrm{L}^2(\Gamma\backslash G)\otimes V_\sigma)^K,$$

where

$${}^{jk}H^\sigma_t(g) = e^{t\lambda_\sigma}\int_K\int_K p_t(k^{-1}gk_1)(\pi_\sigma(kk_1^{-1}))_{jk}\,dk_1\,dk.$$

Thus, $e^{-t\Delta_\sigma}$ is given by the matrix of convolution operators $\pi_\Gamma({}^{jk}H^\sigma_t)$. The kernels ${}^{jk}H^\sigma_t$ are essentially of the same form as the kernels $H^\sigma_{p_t}$ defined in (15), and we arrive at

Theorem 5. *Let $\sigma\in\widehat{K}$, and Δ_σ be the Bochner-Laplace operator on the homogeneous vector bundle $E_\sigma=\Gamma\backslash(G\times V_\sigma)/K\to\Gamma\backslash G/K$. Then*

$$\mathrm{tr}\, e^{-t\Delta_\sigma} = \frac{e^{t\lambda_\sigma}\int_{\mathbb{H}}\mathrm{tr}\,\pi_\sigma(kk_1^{-1})dk_1dk}{(2\pi)^{\dim G/K}t^{\frac{\dim G/K}{2}}}\widetilde{vol}(\Xi/\mathbb{K}) + O(e^{t\lambda_\sigma}t^{-(\dim G/K-1)/2}(\log t)^{\Lambda-1}),$$

where

$$\widetilde{vol}(\Xi/\mathbb{K}) = \sum_\iota\int_{\mathrm{Reg}\,\Xi}\hat{\mathcal{F}}_\iota(p,\xi)\frac{\alpha_\iota(p)j_\iota(p)d(\mathrm{Reg}\,\Xi)(p,\xi)}{vol\,\mathcal{O}_{(p,\xi)}},$$

and $\hat{\mathcal{F}}_\iota(p,\xi) = (4\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i\sum_{l,j}c^j_l(p)\zeta_l\xi_j}e^{-|e^{\sum\zeta_iX_i}|^2/4}\Psi^*(d_G)(\zeta)$, $b>0$, the notation being the same as in Corollary 5.

Proof. By Theorem 4, and (26), we have

$$\operatorname{tr}\pi_\Gamma({}^{jk}H_t^\sigma) = \frac{e^{t\lambda_\sigma}}{(2\pi)^{\dim G/K} t^{\frac{\dim G/K}{2}}} \sum_\iota \int_{\operatorname{Reg}\mathcal{C}} \hat{\mathcal{F}}_\iota(p,\xi) \frac{(\pi_\sigma(kk_1^{-1}))_{jk}\alpha_\iota(p) j_\iota(p) d(\operatorname{Reg}\mathcal{C})(p,\xi,k_1,k)}{|\det \Phi''_{\iota\iota}(p,\xi,k_1,k)_{N_{(p,\xi,k_1,k)}\operatorname{Reg}\mathcal{C}}|^{1/2}}$$

up to terms of order $O(e^{t\lambda_\sigma} t^{-(\dim G/K-1)/2}(\log t)^{\Lambda-1})$. The assertion now follows with (27). □

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