PSEUDODIFFERENTIAL OPERATORS ON PREHOMOGENEOUS VECTOR SPACES

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Abstract. Let G be a connected, linear algebraic group defined over $\mathbb{R}$, acting regularly on a finite dimensional vector space V over $\mathbb{C}$ with $\mathbb{R}$-structure $V_{\mathbb{R}}$. Assume that V possesses a Zariski-dense orbit, so that (G, ϱ, V) becomes a prehomogeneous vector space over $\mathbb{R}$. We consider the left regular representation π of the group of $\mathbb{R}$-rational points $G_{\mathbb{R}}$ on the Banach space $\mathrm{C}_0(V_{\mathbb{R}})$ of continuous functions on $V_{\mathbb{R}}$ vanishing at infinity, and study the convolution operators $\pi(f)$, where f is a rapidly decreasing function on the identity component of $G_{\mathbb{R}}$. Denote the complement of the dense orbit by S, and put $S_{\mathbb{R}} = S \cap V_{\mathbb{R}}$. It turns out that, on $V_{\mathbb{R}} - S_{\mathbb{R}}$, $\pi(f)$ is a smooth operator. If $S_{\mathbb{R}} = \{0\}$, the restriction of the Schwartz kernel of $\pi(f)$ to the diagonal defines a homogeneous distribution on $V_{\mathbb{R}} - \{0\}$. Its non-unique extension to $V_{\mathbb{R}}$ can then be regarded as a trace of $\pi(f)$. If G is reductive, and S and $S_{\mathbb{R}}$ are irreducible hypersurfaces, $\pi(f)$ corresponds, on each connected component of $V_{\mathbb{R}} - S_{\mathbb{R}}$, to a totally characteristic pseudodifferential operator. In this case, the restriction of the Schwartz kernel of $\pi(f)$ to the diagonal defines a distribution on $V_{\mathbb{R}} - S_{\mathbb{R}}$ given by some power $|p(m)|^s$ of a relative invariant $p(m)$ of (G, ϱ, V) and, as a consequence of the Fundamental Theorem of Prehomogeneous Vector Spaces, its extension to $V_{\mathbb{R}}$, and the complex s-plane, satisfies functional equations similar to those for local zeta functions. A trace of $\pi(f)$ can then be defined by subtracting the singular contributions of the poles of the meromorphic extension.

1. Introduction

Let G be a real reductive algebraic group acting on a smooth affine algebraic variety M, and π a representation of G on some suitable function space over M. The purpose of this paper is to understand this representation in case that G acts on M with an open orbit. If π is an irreducible unitary Hilbert representation, it was shown by Harish-Chandra that the convolution operators

$$\pi(f) = \int_G f(g)\pi(g)d_G(g), \tag{1}$$

where f is a smooth, compactly supported function on G, and d_G Haar measure on G, are of trace class, and the global character of π is defined as the distribution $\Theta_\pi : f \to \mathrm{Tr}\,\pi(f)$. It determines π up to unitary equivalence [14]. Our interest will be directed towards the regular representation of G on the Banach space $\mathrm{C}_0(M)$ of continuous function on M vanishing at infinity, and we will mainly be concerned with the microlocal structure of the operators $\pi(f)$ in the general case when the underlying G-action on M fails to be transitive. The operators $\pi(f)$ are then no longer smooth, and the orbit structure of G is reflected in the singular behaviour of the Schwartz kernel of $\pi(f)$, and its restriction to the diagonal. We will work

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in a C^∞ framework, relying on the theory of pseudodifferential operators, and avoid Hilbert space theory entirely. Since the examined varieties are naturally isomorphic to homogeneous vector bundles $G\times_H V$, where H is a reductive subgroup of G and V a locally homogeneous H-module, we can restrict our investigation to the study of locally transitive linear actions of connected reductive groups (see [9], page 214).

We give now a summary of the results obtained in this paper. Let G be a connected, linear algebraic group defined over $\mathbb{R}$, V a n-dimensional vector space over $\mathbb{C}$ with $\mathbb{R}$-structure $V_\mathbb{R}$, and $\varrho : G \to \mathrm{GL}(V)$ a $\mathbb{R}$-rational representation of G on V with a Zariski-dense G-orbit. The triple (G,ϱ,V) constitutes a prehomogeneous vector space, and we denote the dual prehomogeneous vector space by (G,ϱ^*,V^*). Let S and S^* be the respective singular sets. Assume that G is reductive, and that there exists an irreducible, homogeneous polynomial p such that $S=\{m\in V\colon p(m)=0\}$. Then S^* is also given as the set of zeros of a homogeneous, irreducible polynomial p^*. The rational functions p and p^* are called relative invariants. Set $S_\mathbb{R}=V_\mathbb{R}\cap S$, $S^*_\mathbb{R}=V^*_\mathbb{R}\cap S^*$, denote by $G_\mathbb{R}$ the group of $\mathbb{R}$-rational points of G, and let $G_0=(G_\mathbb{R})^0\subset G_\mathbb{R}$ be the connected component containing the unit element. $V_\mathbb{R}-S_\mathbb{R}$ and $V^*_\mathbb{R}-S^*_\mathbb{R}$ decompose into the same number of connected components V_i, respectively V_i^*, $i=1,\dots,l$, each of them being a G_0-orbit. Under these assumptions, the Fundamental Theorem of Prehomogeneous Vector Spaces of Sato states that for rapidly decreasing functions φ, φ^* on $V_\mathbb{R}$, respectively $V^*_\mathbb{R}$, the integrals

$$\int_{V_i}|p(m)|^s\varphi(m)\,dm, \qquad \int_{V_i^*}|p^*(\xi)|^s\varphi^*(\xi)\,d\xi, \tag{2}$$

which converge for $\mathrm{Re}\,s>0$, can be extended analytically to meromorphic functions on the whole complex s-plane, and satisfy the functional equations

$$\int_{V_i^*}|p^*(\xi)|^{s-\frac{n}{\deg p}}\widehat{\varphi}(\xi)d\xi=\gamma\Big(s-\frac{n}{\deg p}\Big)\sum_{i=1}^{l}c_{ij}(s)\int_{V_i}|p(m)|^{-s}\varphi(m)\,dm, \tag{3}$$

where $\gamma(s)$ is given by a product of Γ-functions, and the c_{ij} are entire functions [5]. Consider now the left regular representation π of the group $G_\mathbb{R}$ on the Banach space $\mathrm{C}_0(V_\mathbb{R})$, and let f be a rapidly decreasing function on G_0. It turns out that for arbitrary prehomogeneous vector spaces, the restriction of $\pi(f)$ to $\mathrm{C}^\infty_\mathrm{c}(V_\mathbb{R}-S_\mathbb{R})$ is a pseudodifferential operator with smooth kernel. If $S_\mathbb{R}=\{0\}$, the restriction of the kernel of $\pi(f)$ to the diagonal defines a homogeneous distribution on $V_\mathbb{R}-\{0\}$ given by an integral of the form (2), where $s=-n$. It can be continued to a meromorphic function on the whole complex s-plane, and extended to a homogeneous distribution on $V_\mathbb{R}$. This extension, which is unique up to a distribution supported at the origin, can then be regarded as a trace of $\pi(f)$. In case that G is reductive, and both S and $S_\mathbb{R}$ are irreducible hypersurfaces, each of the components V_i is a C^∞ manifold with boundary ∂V_i, the latter being smooth outside its intersection with the set of non-regular points $S^{\mathrm{sing}}_\mathbb{R}$ of $S_\mathbb{R}$. Assuming that $G_\mathbb{R}(p)=\{g\in G_\mathbb{R}: p(gm)=p(m)\}$ acts locally transitively on the components of $S_\mathbb{R}-S^{\mathrm{sing}}_\mathbb{R}$, we show that the restriction of $\pi(f)$ to $\overline{V}_i-(\partial V_i)^{\mathrm{sing}}$ is a totally characteristic pseudodifferential operator of the class $\mathrm{L}^{-\infty}_b(\overline{V}_i-(\partial V_i)^{\mathrm{sing}})$. These operators were first studied by Melrose, and are locally given as oscillatory integrals of the form

$$Au(x)=\int e^{i(x-y)\cdot\xi}a(x,\xi)u(y)\,dy\,đ\xi,$$

where $a(x,\xi)=\tilde a(x,x_1\xi_1,\xi')$, and $\tilde a\in \mathrm{S}^{-\infty}_{\mathrm{la}}(Z\times\mathbb{R}^n)$ is a lacunary symbol [7]. Here $x=(x_1,x')$ are the standard coordinates in $Z=\overline{\mathbb{R}^+}\times\mathbb{R}^{n-1}$. The kernels of such operators are no longer smooth, and become singular along $S_\mathbb{R} \times S_\mathbb{R}$. If a smooth operator acts on a C^∞ manifold, a trace can be defined in a natural way by restricting its kernel to the diagonal, yielding a density on the underlying manifold. If the manifold is compact, it can be integrated, and the so defined trace coincides with the usual L^2-trace. In our situation, the restriction of the Schwartz kernel of $\pi(f)$ to the diagonal defines a distribution on $V_\mathbb{R} - S_\mathbb{R}$ which is locally given by expressions of the form (2) for some critical exponent s, and its extension to $V_\mathbb{R} - S_\mathbb{R}^{\mathrm{sing}}$, and the complex s-plane, satisfies functional equations according to (3). Again, a trace of $\pi(f)$ can be defined by subtracting the singular contributions of the poles of the meromorphic extension. This trace is closely related to the b-trace introduced by Melrose in his work on totally characteristic pseudodifferential operators.

As an application, we consider the holomorphic semigroup S_t of bounded linear operators on $\mathrm{C}_0(V_\mathbb{R})$ generated by the Casimir element of the representation π. The latter is a differential operator of Euler type, and the corresponding semigroup can be characterized by a convolution semigroup of complex measures which are absolutely continuous with respect to Haar measure. Denoting the corresponding Radon-Nikodym derivative by $K_t(g) \in \mathrm{L}^1(G_\mathbb{R}, d_{G_\mathbb{R}})$, one has $S_t = \pi(K_t)$, and since $K_t(g)$ is analytic in t and g, as well as rapidly decreasing on G_0, we can apply the above considerations (see [11], pages 152 and 209). Strongly elliptic differential operators associated with general Banach representations, and the holomorphic semigroups generated by them, were first studied by Langlands; it was in fact the study of the holomorphic semigroup S_t which originally motivated this work. In particular, we get explicit expressions for the Schwartz kernel of the operators $S_t : \mathrm{C}_\mathrm{c}^\infty(V_\mathbb{R}) \to \mathcal{D}'(V_\mathbb{R})$, and their restrictions to the diagonal.

2. Linear algebraic groups and prehomogeneous vector spaces

Let $G \subset \mathrm{GL}(m, \mathbb{C})$ be a connected, linear algebraic group defined over $\mathbb{R}$, and $G_\mathbb{R} = G \cap \mathrm{GL}(m, \mathbb{R})$ the group of $\mathbb{R}$-rational points of G. We will assume that G is symmetric, so that $G_\mathbb{R}$ becomes a real reductive algebraic group in the sense of [13]. Let $\mathfrak{g}$ be the Lie algebra of $G_\mathbb{R}$, K a maximal compact subgroup of $G_\mathbb{R}$ with Lie algebra $\mathfrak{k}$, and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

the corresponding Cartan decomposition of $\mathfrak{g}$. The mapping $(k, X) \mapsto k \exp X$ represents an analytic diffeomorphism of $K \times \mathfrak{p}$ onto $G_\mathbb{R}$, while $\exp : \mathfrak{k} \to K$ is a surjection. Let G_0 be the connected component of $G_\mathbb{R}$ containing the unit element e. Clearly, G_0 is a Lie group, and its Lie algebra coincides with $\mathfrak{g}$. Choose a left invariant Riemannian metric on G_0, and denote the distance of two points $g, h \in G_0$ by $d(g, h)$. We set $|g| = d(g, e)$. Then

$$|g| = |g^{-1}|, \qquad |e| = 0, \qquad |gh| \leq |g| + |h|, \qquad g, h \in G_0.$$

In the following, we will say that a function f on G_0 is *at most of exponential growth*, if there exists a $\kappa > 0$ such that $|f(g)| \leq Ce^{\kappa|g|}$ for some constant $C > 0$. If $g = k \exp X$ is the Cartan decomposition of an arbitrary element in G_0, one computes

$$|g| \leq C_K + |\exp X|, \qquad |\exp X| \leq |k| + |g| \leq C_K + |g|,$$

where $C_K = \max_{k \in K} |k| < \infty$. Putting $C' = e^{C_K} \geq 1$, we obtain

$$\frac{1}{C'} e^{|\exp X|} \leq e^{|g|} \leq C' e^{|\exp X|}. \tag{4}$$

Let d be the dimension of $G_\mathbb{R}$, $X_1, \dots, X_l$ a basis of $\mathfrak{k}$ and $X_{l+1}, \dots, X_d$ a basis of $\mathfrak{p}$. Since $\exp : \mathfrak{p} \simeq P$ is an analytic diffeomorphism, there exists a $C'' \geq 1$, such that

$$\frac{1}{C''} |X| \leq |\exp X| \leq C'' |X|, \tag{5}$$

where $|X| = \sqrt{q_{l+1}^2 + \cdots + q_d^2}$ is the length of $X = q_{l+1}X_{l+1} + \cdots + q_d X_d \in \mathfrak{p}$. Realizing exp as power series for matrices, relations (4) and (5) imply that the matrix coefficients of $\exp X$ and, consequently, of $g = k \exp X$, are at most of exponential growth.

Let L and R be the left respectively right regular representation of G_0 on the Fréchet space $\mathrm{C}^\infty(G_0)$ of smooth, complex valued functions on G_0, equipped with the usual topology of uniform convergence on compact subsets of a function and each of its derivatives (see [14], page 220). The corresponding representations on $\mathrm{C}^\infty(G_0)$ of the universal enveloping algebra $\mathfrak{U}$ of the complexification of $\mathfrak{g}$ will be written dL respectively dR. As before, d_{G_0} stands for left invariant Haar measure on G_0. We introduce now the space of *rapidly decreasing functions on* G_0 (compare also [13], page 230). Its definition was originally motivated by the decay properties of the kernel $K_t(g)$ of a holomorphic semigroup S_t generated by a strongly elliptic differential operator associated with some Banach representation. This will be explained in more detail in Section 7.

Definition 1. *The space of* rapidly decreasing functions on G_0, *in the following denoted by $\mathcal{S}(G_0)$, is given by all functions $f \in \mathrm{C}^\infty(G_0)$ satisfying the following conditions:*
(i) For every $\kappa \geq 0$, and $X \in \mathfrak{U}$, there exists a constant C such that

$$|dL(X)f(g)| \leq Ce^{-\kappa|g|};$$

(ii) for every $\kappa \geq 0$, and $X \in \mathfrak{U}$, one has $dL(X)f \in \mathrm{L}^1(G_0, e^{\kappa|g|}d_{G_0})$.

Note that $\mathcal{S}(G_0)$ is $\pi(G_0)$- and $d\pi(\mathfrak{U})$-invariant.

Remark 1. Let $f \in \mathcal{S}(G_0)$. Since $g\,\mathrm{e}^X g^{-1} = \mathrm{e}^{\mathrm{Ad}\,(g)X}$, one calculates, with respect to the basis of $\mathfrak{g}$ introduced above,

$$\begin{aligned} dR(X_i)f(g) &= \lim_{h\to 0} h^{-1}[f(g\,\mathrm{e}^{hX_i}\,g^{-1}g) - f(g)] = -dL(\mathrm{Ad}\,(g)X_i)f(g) \\ &= -\sum_{j=1}^d A_{ij}(g)dL(X_j)f(g), \end{aligned}$$

where the matrix coefficients A_{ij} of the adjoint representation, and their derivatives, are at most of exponential growth. Thus, by the Theorem of Poincaré-Birkoff-Witt, $dR(X)f$ satisfies the conditions (i) and (ii) of the preceeding definition for arbitrary $X \in \mathfrak{U}$, and $\kappa \geq 0$.

In the sequel, the following partial integration formulas will be needed. Let $\Delta(g) = |\det \mathrm{Ad}\,(g)|$ be the modular function of G_0.

Proposition 1. *Let $f_1 \in \mathcal{S}(G_0)$, and assume that $f_2 \in \mathrm{C}^\infty(G_0)$, together with all its derivatives, is at most of exponential growth. Then, for arbitrary multiindices γ,*

$$\int_{G_0} f_1(g)dL(X^\gamma)f_2(g)d_{G_0}(g) = (-1)^{|\gamma|}\int_{G_0} dL(X^{\tilde\gamma})f_1(g)f_2(g)d_{G_0}(g), \tag{6}$$

$$\int_{G_0} f_1(g)dR(X^\gamma)f_2(g)d_{G_0}(g) = \sum_{\alpha_1+\alpha_2=\gamma} \iota^{\alpha_1}(-1)^{|\alpha_2|}\int_{G_0} dR(X^{\tilde\alpha_2})f_1(g)f_2(g)d_{G_0}(g), \tag{7}$$

where $X^\gamma = X_{i_1}^{\gamma_1}\dots X_{i_r}^{\gamma_r}$, $X^{\tilde\gamma} = X_{i_r}^{\gamma_r}\dots X_{i_1}^{\gamma_1}$, and $\iota_k = dR(X_k)\Delta(e) = dL(-X_k)\Delta(e)$.

Proof. First, note that $f_1(g)f_2(\mathrm{e}^{-hX_i}g)$ is a differentiable function with respect to h, and integrable on G_0 for all $h \in \mathbb{R}$, since $|f_1(g)f_2(\mathrm{e}^{-hX_i}g)| \leq C|f_1(g)|e^{\kappa(|\mathrm{e}^{hX_i}|+|g|)}$ for some

$C, \kappa > 0$, and $f_1(g) \in \mathrm{L}^1(G_0, e^{\kappa|g|} d_{G_0})$. Furthermore,

$$\frac{d}{dh} f_2(\mathrm{e}^{-hX_i}\, g)_{|h=h_0} = \frac{d}{dh} f_2(\mathrm{e}^{-hX_i}\, \mathrm{e}^{-h_0 X_i}\, g)_{|h=0} = \pi(\mathrm{e}^{h_0 X_i}\,) dL(X_i) f_2(g),$$

so that, for $h_0 \in \mathbb{R}$,

$$\left| f_1(g) \frac{d}{dh} f_2(\mathrm{e}^{-hX_i}\, g)_{|h=h_0} \right| = |f_1(g) \pi(\mathrm{e}^{h_0 X_i}\,) dL(X_i) f_2(g)| \leq C' |f_1(g)| e^{\kappa'(|\,\mathrm{e}^{h_0 X_i}\,|+|g|)},$$

where $C', \kappa' > 0$. Thus, for $h_0 \in (-\varepsilon, \varepsilon)$, $f_1(g) \frac{d}{dh} f_2(\mathrm{e}^{-hX_i}\, g)_{|h=h_0}$ can be estimated from above by $C_\varepsilon |f_1(g)| e^{\kappa'|g|}$, $C_\varepsilon > 0$ being some constant. By the Lebesgue Theorem on Dominated Convergence, and the left invariance of Haar measure, we therefore obtain

$$\int_{G_0} f_1(g) dL(X_i) f_2(g) d_{G_0}(g) = \int_{G_0} \lim_{h \to 0} h^{-1} [f_1(g)(f_2(\mathrm{e}^{-hX_i}\, g) - f_2(g))] d_{G_0}(g)$$

$$= \lim_{h \to 0} h^{-1} \int_{G_0} f_1(g)[f_2(\mathrm{e}^{-hX_i}\, g) - f_2(g)] d_{G_0}(g) = \lim_{h \to 0} h^{-1} \int_{G_0} [f_1(\mathrm{e}^{hX_i}\, g) - f_1(g)] f_2(g) d_{G_0}(g),$$

compare [1], pages 146 and 364. Since $f_1(\mathrm{e}^{hX_i}\, g) f_2(g)$ and $f_2(g) \frac{d}{dh} f_1(\mathrm{e}^{hX_i}\, g)_{|h=h_0}$, $h_0 \in (-\varepsilon, \varepsilon)$ can be estimated in a similar way, a repeated application of the Lebesgue Theorem finally yields (6) for $|\gamma| = 1$. Similarly, by taking into account Remark 1, we deduce

$$\int_{G_0} f_1(g) dR(X_i) f_2(g) d_{G_0}(g) = \lim_{h \to 0} h^{-1} \int_{G_0} f_1(g)[f_2(g\,\mathrm{e}^{hX_i}\,) - f_2(g)] d_{G_0}(g)$$

$$= \lim_{h \to 0} h^{-1} \int_{G_0} [f_1(g\,\mathrm{e}^{-hX_i}\,) \Delta(\mathrm{e}^{-hX_i}\,)^{-1} - f_1(g)] f_2(g) d_{G_0}(g),$$

obtaining (7) for $|\gamma| = 1$. The general formulas then follow by induction. □

In what follows, we will denote the coordinate functions in $M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ by g_{ij}. With the identification $\mathfrak{g} \simeq \mathbb{R}^d$, and with respect to a basis $X_1, \dots, X_d$ of $\mathfrak{g}$, the canonical coordinates of second type of a point $g \in G_\mathbb{R}$ are given by

$$\Phi_g : gU_e \ni g\,\mathrm{e}^{\zeta_1 X_1} \dots \mathrm{e}^{\zeta_d X_d} \mapsto (\zeta_1, \dots, \zeta_d) \in W_0, \tag{8}$$

where W_0 denotes a sufficiently small neighbourhood of 0 in $\mathbb{R}^d$, and $U_e = \exp(W_0)$. We will write Φ for Φ_e. As a real analytic submanifold of $\mathrm{GL}(n, \mathbb{R})$, the group of $\mathbb{R}$-rational points $G_\mathbb{R}$ is embedded in $M_n(\mathbb{R})$, meaning that the matrix

$$D\Phi_g^{-1}(\zeta) = \left(\frac{d}{d\zeta_k} (g_{ij} \circ \Phi_g^{-1})(\zeta) \right)_{ij,k}$$

of the derivative of Φ_g^{-1} has rang d for all $g \in G_\mathbb{R}$ and $\zeta \in W_0$. If we therefore define for each tuple $i_1 j_1, \dots, i_d j_d$ of indices the sets

$$G_\mathbb{R}^{i_1 j_1, \dots, i_d j_d} = \left\{ g \in G_\mathbb{R} : \frac{\partial(g_{i_1 j_1} \circ \Phi_g^{-1}, \dots, g_{i_d j_d} \circ \Phi_g^{-1})}{\partial(\zeta_1, \dots, \zeta_d)} \neq 0 \right\}, \tag{9}$$

we get a finite covering of $G_\mathbb{R}$ by open, although not necessarily connected, sets.

We will shortly review some topics in the theory of prehomogeneous vector spaces. Let V be a n-dimensional vector space over $\mathbb{C}$ with $\mathbb{R}$-structure $V_\mathbb{R}$, $\varrho : G \to \mathrm{GL}(V)$ a $\mathbb{R}$-rational representation of G on V, and suppose that ϱ has a dense G-orbit in the Zariski topology. Then (G, ϱ, V) is called a *prehomogeneous vector space over* $\mathbb{R}$. The complement set of the unique dense orbit, the *singular set*, is a Zariski closed set, and will be denoted by S. Since $\varrho(G_R) \subset \varrho(G)_\mathbb{R}$, the restriction of ϱ to $G_\mathbb{R}$ induces a regular $G_\mathbb{R}$-action on $V_\mathbb{R}$. Assume now that there is an irreducible, homogeneous polynomial p such that $S = \{m \in V : p(m) = 0\}$.

It is called a *relative invariant*, and there exists a rational character $\chi : G \to \mathrm{GL}(1, \mathbb{C})$ satisfying

$$p(\varrho(g)m)) = \chi(g)p(m), \qquad m \in V - S, \quad g \in G. \tag{10}$$

The symmetry of G implies that G is reductive, so that if S is an irreducible hypersurface, (G, ϱ, V) is a regular prehomogeneous vector space, and one has $\deg p | 2n$, $\det \varrho(g)^2 = \chi(g)^{2n/\deg p}$, where $n = \dim_{\mathbb{C}} V$. By multiplying p with a scalar, we can always assume that $p(V_{\mathbb{R}}) \subset \mathbb{R}$, and $\chi(G_{\mathbb{R}}) \subset \mathbb{R}^*$. Let (G, ϱ^*, V^*) denote the dual prehomogeneous vector space of (G, ϱ, V); then its singular set S^* is also given as the set of zeros of a relative invariant p^* satisfying $p^*(\varrho^*(g)\xi) = \chi(g)^{-1}p^*(\xi)$, and again we can assume $p^*(V^*_{\mathbb{R}}) \subset \mathbb{R}$. Put $S_{\mathbb{R}} = V_{\mathbb{R}} \cap S, S^*_{\mathbb{R}} = V^*_{\mathbb{R}} \cap S^*$. Then, by a theorem of Whitney, $V_{\mathbb{R}} - S_{\mathbb{R}}$ and $V^*_{\mathbb{R}} - S^*_{\mathbb{R}}$ decompose into the same finite number of connected components, each of them being a G_0-orbit, and we denote them by V_i, respectively V^*_i, $i = 1, \dots, l$. In what follows, we will identify V with $\mathbb{C}^n$, and $V_{\mathbb{R}}$ with $\mathbb{R}^n$, by choosing a basis in $V_{\mathbb{R}}$, and assume that $G \subset \mathrm{GL}(n, \mathbb{C})$ without loss of generality. Note that $n \leq d = \dim G$. Instead of $\varrho(g)m$ we will then simply write gm. In particular, S and $S_{\mathbb{R}}$ become irreducible affine algebraic varieties in $\mathbb{C}^n$, respectively $\mathbb{R}^n$. Their sets of regular points, S^{reg}, respectively $S^{\mathrm{reg}}_{\mathbb{R}}$ can be provided with differentiable structures, the underlying topology being the induced one. Let $\mathcal{S}(V_{\mathbb{R}})$, respectively $\mathcal{S}(V^*_{\mathbb{R}})$, denote the Schwartz space of functions on $V_{\mathbb{R}}$, respectively $V^*_{\mathbb{R}}$, and let $\hat{\varphi}$ be the Fourier transform of $\varphi \in \mathcal{S}(V_{\mathbb{R}})$. Let dm be Lebesgue measure on $V_{\mathbb{R}}$. The following result, regarding the Fourier transform of a complex power of a relative invariant, is known as the Fundamental Theorem of Prehomogeneous Vector Spaces, and was proved by Sato in 1961 (see [5], page 124).

Theorem 1 (Sato). *Let $\varphi \in \mathcal{S}(V_{\mathbb{R}})$ and $\varphi^* \in \mathcal{S}(V^*_{\mathbb{R}})$. Then the integrals*

$$F_j(s, \varphi) = \frac{1}{\gamma(s)} \int_{V_j} |p(m)|^s \varphi(m) dm, \qquad F^*_i(s, \varphi) = \frac{1}{\gamma(s)} \int_{V^*_i} |p^*(\xi)|^s \varphi^*(\xi) d\xi, \tag{11}$$

converge for $\mathrm{Re}\, s > 0$, *and can be extended analytically to holomorphic functions on the whole s-plane, satisfying the functional equations*

$$F^*_i(s - n/\deg p, \hat{\varphi}) = \gamma(-s) \sum_{j=1}^{l} c_{ij}(s) F_j(-s, \varphi). \tag{12}$$

Here $\gamma(s)$ is given by a product of Γ-functions, and the $c_{ij}(s)$ are entire functions which do not depend on φ. The functions $F_j(s, \varphi)$, $F^*_j(s, \varphi)$ are called local ζ-functions.

Originally, the theory of prehomogeneous vector spaces developed from an attempt to construct Dirichlet series satisfying functional equations in a systematic way. For a detailed exposition the reader is referred to [5].

3. Review of pseudodifferential operators

3.1. **Generalities.** This section is devoted to the exposition of some of the basic facts about pseudodifferential operators, needed to formulate our main results in the sequel. Our main references for the theory will be [4] and [12]. Consider first an open set U in $\mathbb{R}^n$, and let $x_1, \dots, x_n$ be the standard coordinates. For any real number l, we denote by $\mathrm{S}^l(U \times \mathbb{R}^n)$ the class of all functions $a(x, \xi) \in \mathrm{C}^\infty(U \times \mathbb{R}^n)$ such that, for any multiindices α, β, and any compact set $K \subset U$, there exist constants $C_{\alpha,\beta,K}$ for which

$$|(\partial^\alpha_\xi \, \partial^\beta_x \, a)(x, \xi)| \leq C_{\alpha,\beta,K} \, \langle \xi \rangle^{l - |\alpha|}, \qquad x \in K, \quad \xi \in \mathbb{R}^n, \tag{13}$$

where $\langle \xi \rangle$ stands for $(1+|\xi|^2)^{1/2}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We further put $\mathrm{S}^{-\infty}(U \times \mathbb{R}^n) = \bigcap_{l \in \mathbb{R}} \mathrm{S}^l(U \times \mathbb{R}^n)$. Note that, in general, the constants $C_{\alpha,\beta,K}$ also depend on $a(x,\xi)$. For any such $a(x,\xi)$ one then defines the continuous linear operator

$$A : \mathrm{C}^\infty_{\mathrm{c}}(U) \longrightarrow \mathrm{C}^\infty(U)$$

by the formula

$$Au(x) = \int e^{ix\cdot\xi} a(x,\xi) \hat{u}(\xi) đ\xi, \tag{14}$$

where $\hat{u}$ denotes the Fourier transform of u, and $đ\xi = (2\pi)^{-n}\, d\xi$. An operator A of this form is called a *pseudodifferential operator*, and we denote the class of all such operators for which $a(x,\xi) \in \mathrm{S}^l(U \times \mathbb{R}^n)$ by $\mathrm{L}^l(U)$. The set $\mathrm{L}^{-\infty}(U) = \bigcap_{l \in \mathbb{R}} \mathrm{L}^l(U)$ consists of all operators with smooth kernel. They are called *smooth operators*. By inserting in (14) the definition of $\hat{u}$, we obtain for Au the expression

$$Au(x) = \int\int e^{i(x-y)\cdot\xi} a(x,\xi) u(y)\, dy\, đ\xi, \tag{15}$$

which exists as an oscillatory integral. The Schwartz kernel $K_A \in \mathcal{D}'(U \times U)$ of A is given by the oscillatory integral

$$K_A(x,y) = \int e^{i(x-y)\cdot\xi} a(x,\xi)\, đ\xi, \tag{16}$$

and is a smooth function outside the diagonal in $U \times U$.

Consider next an n-dimensional C^∞ manifold X, and let $(\kappa_\gamma, U^\gamma)$ be an atlas for X. Then a linear operator

$$A : \mathrm{C}^\infty_{\mathrm{c}}(X) \longrightarrow \mathrm{C}^\infty(X) \tag{17}$$

is called a *pseudodifferential operator on X of order l* if for every chart diffeomorphism $\kappa_\gamma : U^\gamma \to \tilde{U}^\gamma = \kappa_\gamma(U^\gamma)$, the operator $A^\gamma u = [A_{|U^\gamma}(u \circ \kappa_\gamma)] \circ \kappa_\gamma^{-1}$ given by the diagram

$$\begin{array}{ccc} \mathrm{C}^\infty_{\mathrm{c}}(U^\gamma) & \xrightarrow{\;A_{|U^\gamma}\;} & \mathrm{C}^\infty(U^\gamma) \\ \uparrow \kappa_\gamma^* & & \uparrow \kappa_\gamma^* \\ \mathrm{C}^\infty_{\mathrm{c}}(\tilde{U}^\gamma) & \xrightarrow{\;A^\gamma\;} & \mathrm{C}^\infty(\tilde{U}^\gamma) \end{array}$$

is a pseudodifferential operator on $\tilde{U}^\gamma$ of order l, and we write $A \in \mathrm{L}^l(X)$. Note that, since the U^γ are not necessarily connected, we can choose them in such a way that $X \times X$ is covered by the open sets $U^\gamma \times U^\gamma$. Now, in general, if X and Y are two smooth manifolds, and

$$A : \mathrm{C}^\infty_{\mathrm{c}}(X) \longrightarrow \mathrm{C}^\infty(Y) \subset \mathcal{D}'(Y)$$

is a continuous linear operator, where $\mathcal{D}'(Y) = (\mathrm{C}^\infty_{\mathrm{c}}(Y,\Omega))'$ and $\Omega = |\Lambda^n(Y)|$ is the density bundle on Y, its Schwartz kernel is given by the distribution section $K_A \in \mathcal{D}'(Y \times X, \mathbf{1} \boxtimes \Omega_X)$, where $\mathcal{D}'(Y \times X, 1 \boxtimes \Omega_X) = (\mathrm{C}^\infty_{\mathrm{c}}(Y \times X, (\mathbf{1} \boxtimes \Omega_X)^* \otimes \Omega_{Y\times X}))'$. Observe that $\mathrm{C}^\infty_{\mathrm{c}}(Y,\Omega_Y) \otimes \mathrm{C}^\infty(X) \simeq \mathrm{C}^\infty(Y \times X, (\mathbf{1} \boxtimes \Omega_X)^* \otimes \Omega_{Y\times X})$. In case that $X = Y$ and $A \in \mathrm{L}^l(X)$, A is given locally by the operators A^γ, which can be written in the form

$$A^\gamma u(x) = \int\int e^{i(x-y)\cdot\xi} a^\gamma(x,\xi) u(y)\, dy đ\xi,$$

where $u \in \mathrm{C}^\infty_c(\tilde U^\gamma)$, $x \in \tilde U^\gamma$, and $a^\gamma(x,\xi) \in \mathrm{S}^l(\tilde U^\gamma, \mathbb{R}^n)$. The kernel of A is then determined by the kernels $K_{A^\gamma} \in \mathcal{D}'(\tilde U^\gamma \times \tilde U^\gamma)$. For $l < -\dim X$, they are continuous, and given by absolutely convergent integrals. In this case, their restrictions to the respective diagonals in $\tilde U^\gamma \times \tilde U^\gamma$ define continuous functions

$$k^\gamma(m) = K_{A^\gamma}(\kappa_\gamma(m), \kappa_\gamma(m)), \qquad m \in U^\gamma,$$

which, for $m \in U^{\gamma_1} \cap U^{\gamma_2}$, satisfy the relations $k^{\gamma_2}(m) = |\det(\kappa_{\gamma_1} \circ \kappa_{\gamma_2}^{-1})'| \circ \kappa_{\gamma_2}(m) k^{\gamma_1}(m)$, thus defining a density $k \in C(X, \Omega)$ on $\Delta_{X\times X} \simeq X$.

3.2. **Totally characteristic pseudodifferential operators.** We introduce now a special class of pseudodifferential operators associated in a natural way to a C^∞ manifold X with boundary ∂X. Our main reference will be [7] in this case. Let $\mathrm{C}^\infty(X)$ be the space of functions on X which are C^∞ up to the boundary, and $\dot{\mathrm{C}}^\infty(X)$ the subspace of functions vanishing to all orders on ∂X. The standard spaces of distributions over X are

$$\mathcal{D}'(X) = (\dot{\mathrm{C}}^\infty_c(X, \Omega))', \qquad \dot{D}'(X) = (\mathrm{C}^\infty_c(X, \Omega))',$$

the first being the space of *extendible distributions*, whereas the second is the space of *distributions supported by* X. Consider now the translated partial Fourier transform of a symbol $a(x,\xi) \in \mathrm{S}^l(\mathbb{R}^n \times \mathbb{R}^n)$,

$$Ma(x, \xi'; t) = \int e^{i(1-t)\xi_1} a(x, \xi_1, \xi') d\xi_1,$$

which is C^∞ away from $t = 1$. Here $\xi = (\xi_1, \xi')$. One says that $a(x,\xi)$ is *lacunary* if it satisfies the *lacunary condition*

$$Ma(x, \xi'; t) = 0 \qquad \text{for } t < 0. \tag{18}$$

The subspace of lacunary symbols will be denoted by $\mathrm{S}^l_{la}(\mathbb{R}^n \times \mathbb{R}^n)$. Let $Z = \overline{\mathbb{R}^+} \times \mathbb{R}^{n-1}$ be the standard manifold with boundary with the natural coordinates $x = (x_1, x')$. In order to define on Z operators of the form (15), where now $a(x,\xi) = \tilde a(x_1, x', x_1\xi_1, \xi')$ is a more general amplitude and $\tilde a(x,\xi)$ is lacunary, one rewrites the formal adjoint of A by making a singular coordinate change. Thus, for $u \in \mathrm{C}^\infty_c(Z)$, one considers

$$A^* u(y) = \int\int e^{i(y-x)\xi} \overline{a}(x,\xi) u(x) \, dx đ\xi.$$

By putting $\lambda = x_1\xi_1$, $s = x_1/y_1$, this can be rewritten as

$$A^* u(y) = (2\pi)^{-n} \int\int\int\int e^{i(1/s-1, y'-x')\cdot(\lambda,\xi')} \overline{\tilde a}(y_1 s, x', \lambda, \xi') u(y_1 s, x') d\lambda \frac{ds}{s} dx' d\xi'. \tag{19}$$

According to [7], Propositions 3.6 and 3.9, for every $\tilde a \in \mathrm{S}^{-\infty}_{la}(Z \times \mathbb{R}^n)$, the successive integrals in (19) converge absolutely and uniformly, thus defining a continuous bilinear form

$$\mathrm{S}^{-\infty}_{la}(Z \times \mathbb{R}^n) \times \mathrm{C}^\infty_c(Z) \longrightarrow \mathrm{C}^\infty(Z),$$

which extends to a separately continuous form

$$\mathrm{S}^{\infty}_{la}(Z \times \mathbb{R}^n) \times \mathrm{C}^\infty_c(Z) \longrightarrow \mathrm{C}^\infty(Z).$$

If $\tilde a \in \mathrm{S}^\infty_{la}(Z \times \mathbb{R}^n)$, one then defines the operator

$$A : \dot{\mathcal{E}}'(Z) \longrightarrow \dot{\mathcal{D}}'(Z), \tag{20}$$

written formally as (15), as the adjoint of A^*. In this way, the oscillatory integral (15) is identified with a separately continuous bilinear mapping

$$\mathrm{S}^\infty_{la}(Z \times \mathbb{R}^n) \times \dot{\mathcal{E}}'(Z) \longrightarrow \dot{\mathcal{D}}'(Z).$$

The space $\mathrm{L}^l_b(Z)$ of *totally characteristic pseudodifferential operators on* Z *of order* l consists of those continuous linear maps (20) such that vAu is of the form (15) with $\tilde{a}(x,\xi) \in \mathrm{S}^l_{la}(Z\times\mathbb{R}^n)$ whenever $u,v \in \mathrm{C}^\infty_c(Z)$. Similarly, a continuous linear map (17) on a smooth manifold X with boundary ∂X is an element of the space $\mathrm{L}^l_b(X)$ of *totally characteristic pseudodifferential operators on* X *of order* l, if for a given atlas $(\kappa_\gamma, U^\gamma)$ the operators $A^\gamma u = [A_{|U^\gamma}(u\circ\kappa_\gamma)]\circ\kappa_\gamma^{-1}$ are elements of $\mathrm{L}^l_b(Z)$, where U^γ are coordinate patches isomorphic to subsets in Z.

4. A structure theorem for prehomogeneous vector spaces

Let (G,ϱ,V) be a prehomogeneous vector space defined over $\mathbb{R}$, and identify V with $\mathbb{C}^n$, and $V_\mathbb{R}$ with $\mathbb{R}^n$, by choosing a basis in $V_\mathbb{R}$. Consider the Banach space $\mathrm{C}_0(V_\mathbb{R})$ of continuous, complex valued functions on $V_\mathbb{R}$ vanishing at infinity, equipped with the supremum norm. Let $(\pi, \mathrm{C}_0(V_\mathbb{R}))$ be the corresponding continuous left regular representation of $G_\mathbb{R}$ [10]. The representation of $\mathfrak{U}$ on the space of differentiable vectors $\mathrm{C}_0(V_\mathbb{R})_\infty$ will be denoted by $d\pi$. We will also consider the left regular representation of $G_\mathbb{R}$ on $\mathrm{C}^\infty(V_\mathbb{R})$ which, equipped with the topology of uniform convergence on compact subsets, becomes a Fréchet space. This representation will be denoted by π as well. As before, we write G_0 for the connected component of $G_\mathbb{R}$ containing the unit element, and d_{G_0} for Haar measure on G_0. As explained in [10], we can associate to every $f \in \mathcal{S}(G_0)$, and $\varphi \in \mathrm{C}_0(V_\mathbb{R})$ the element $\int_{G_0} f(g)\pi(g)\varphi\, d_{G_0}(g) \in \mathrm{C}_0(V_\mathbb{R})$ which is defined as a Bochner integral; the continuous linear operator on $\mathrm{C}_0(V_\mathbb{R})$ obtained this way is denoted by (1). Its restriction to $\mathrm{C}^\infty_c(V_\mathbb{R})$ induces a continuous linear operator

$$\pi(f) : \mathrm{C}^\infty_c(V_\mathbb{R}) \longrightarrow \mathrm{C}_0(V_\mathbb{R}) \subset \mathcal{D}'(V_\mathbb{R}). \tag{21}$$

To see this, let $u_j \in \mathrm{C}^\infty_c(V_\mathbb{R})$ be a sequence which converges to zero in $\mathrm{C}^\infty_c(V_\mathbb{R})$, that is, assume that there exists a compact set $K \subset V_\mathbb{R}$ such that $\mathrm{supp}\, u_j \subset K$, and $\sup|\,\partial^\alpha u_j| \to 0$ for all j and arbitrary multiindices α. Then, for $v \in \mathrm{C}^\infty_c(V_\mathbb{R})$, one has

$$|\,\langle \pi(f)u_j, v\rangle\,| = \left|\int_{V_\mathbb{R}} (\pi(f)u_j)(m)v(m)dm\right| \leq \|\pi(f)u_j\| \cdot \|v\|_{L^1} \to 0,$$

since $\|\pi(f)u_j\| \leq \|\pi(f)\|\,\|u_j\|$, and $\|u_j\| \to 0$ by assumption. According to Schwartz, there exists a distribution $\mathcal{K}_f \in \mathcal{D}'(V_\mathbb{R}\times V_\mathbb{R})$ such that $\langle\pi(f)u, v\rangle = \mathcal{K}_f(v\otimes u)$ for all $u,v \in \mathrm{C}^\infty_c(V_\mathbb{R})$. The properties of the Schwartz kernel $\mathcal{K}_f$ will depend on the analytic properties of f, as well as the orbit structure of the underlying G_0-action, and our main effort in the following sections will be directed towards the elucidation of the structure of $\mathcal{K}_f$. Denote the coordinates of a point in $V_\mathbb{R}$, respectively $V^*_\mathbb{R}$, by $m_1,\dots,m_n$, respectively $\xi_1,\dots,\xi_n$, and put

$$\varphi_\xi(m) = e^{im\cdot\xi} = e^{i\sum_{j=1}^n m_j\xi_j}, \qquad m \in V_\mathbb{R},\, \xi \in V^*_\mathbb{R}.$$

One has $\varphi_\xi(m) \in \mathrm{C}^\infty(V_\mathbb{R}\times V^*_\mathbb{R})$, and since $f \in \mathrm{L}^1(G_0, d_{G_0})$, we can define the function

$$\hat{f}(m,\xi) = \int_{G_0} f(g)(\pi(g)\varphi_\xi)(m)d_{G_0}(g) = \int_{G_0} f(g)e^{i(g^{-1}m)\cdot\xi}d_{G_0}(g), \quad m \in V_\mathbb{R},\, \xi\in V^*_\mathbb{R}. \tag{22}$$

It is continuous in m and ξ, and $|\hat{f}(m,\xi)| \leq \|f\|_{\mathrm{L}^1}$. Since $(\partial^\alpha_\xi\,\partial^\beta_m\,\pi(g)\varphi_\xi)(m)$ is, together with all its derivatives, at most of exponential growth in g, we can interchange the order of integration and differentiation in (22), yielding $\hat{f}(m,\xi) \in \mathrm{C}^\infty(V_\mathbb{R}\times\mathbb{R}^n_\xi)$. The main issue of this section will consist in proving the theorem.

Theorem 2 (Structure Theorem)**.** *Let (G, ϱ, V) be a prehomogeneous vector space over $\mathbb{R}$, and $(\pi, \mathrm{C}_0(V_R))$ the left regular representation of $G_{\mathbb{R}}$ on $V_{\mathbb{R}}$. Then, for $f \in \mathcal{S}(G_0)$, $\pi(f) : \mathrm{C}^\infty_\mathrm{c}(V_{\mathbb{R}}) \to \mathcal{D}'(V_{\mathbb{R}} - S_{\mathbb{R}})$ is a Fourier integral operator*

$$(\pi(f)\varphi)(m) = \int\int e^{i(m-m')\cdot\xi} a_f(m,\xi)\varphi(m')\, dm'\, d\xi, \tag{23}$$

whose symbol is given by

$$a_f(m,\xi) = e^{-im\cdot\xi}\hat{f}(m,\xi) \in \mathrm{S}^{-\infty}((V_{\mathbb{R}} - S_{\mathbb{R}}) \times V^*_{\mathbb{R}}).$$

On $V_{\mathbb{R}} - S_{\mathbb{R}}$, $\pi(f)$ is a pseudodifferential operator of class $\mathrm{L}^{-\infty}(V_{\mathbb{R}} - S_{\mathbb{R}})$.

For the proof, we will need some lemmas. First note that on $V_{\mathbb{R}} - S_{\mathbb{R}}$, one can express the canonical vector fields ∂_{m_i} of $V_{\mathbb{R}}$ locally as linear combinations in the fundamental vector fields of the underlying $G_{\mathbb{R}}$-action.

Lemma 1. *Let $\tilde{m} \in V_{\mathbb{R}} - S_{\mathbb{R}}$. Then there exists a neighbourhood $U \subset V_{\mathbb{R}} - S_{\mathbb{R}}$ of $\tilde{m}$, and rational functions $\Theta^k_i(m)$ on U, such that, for arbitrary $\varphi \in \mathrm{C}^\infty(V_{\mathbb{R}})$,*

$$(\partial_{m_i}\varphi)(m) = \sum_{k=1}^n \Theta^k_i(m) d\pi(X_{j_k}\varphi)(m), \qquad m \in U, \tag{24}$$

for some indices $j_1, \dots, j_n$.

Proof. For $m \in V_{\mathbb{R}}$, one has

$$d\pi(X_j)\varphi(m) = \frac{d}{dh}\varphi(e^{-hX_j}m)_{|_{h=0}} = \sum_{i=1}^n \frac{d}{dh} m_i(e^{-hX_j}m)_{|_{h=0}}(\partial_{m_i}\varphi)(m). \tag{25}$$

Because of the local transitivity of the $G_{\mathbb{R}}$-action on $V_{\mathbb{R}} - S_{\mathbb{R}}$, the $n \times d$-matrix

$$\Big(\frac{d}{dh} m_i(e^{-hX_j}\tilde{m})_{|_{h=0}}\Big)_{i,j} = \Big(d\pi(X_j)m_i(\tilde{m})\Big)_{i,j}$$

has maximal rang n for $\tilde{m} \in V_{\mathbb{R}} - S_{\mathbb{R}}$, where, by assumption, $d \geq n$. For this reason, there exists a neighbourhood U of $\tilde{m}$, and indices $j_1, \dots, j_n$, such that for all $m \in U$

$$\det\left(\frac{d}{dh} m_i(e^{-hX_{j_k}}m)_{|_{h=0}}\right)_{i,k} \neq 0.$$

Denoting the matrix coefficients of the corresponding inverse matrix by $\Theta^k_i(m)$, the assertion follows with (25). $\square$

Consider the covering of $G_{\mathbb{R}}$ by the sets $G^{i_1 j_1, \dots, i_d j_d}_{\mathbb{R}}$ introduced in (9), and put $G^{i_1 j_1, \dots i_d j_d}_0 = G^{i_1 j_1, \dots, i_d j_d}_{\mathbb{R}} \cap G_0$.

Proposition 2. *Fix $\tilde{m} \in V_{\mathbb{R}} - S_{\mathbb{R}}$. Then there exists a neighbourhood $U \subset V_{\mathbb{R}} - S_{\mathbb{R}}$ of $\tilde{m}$ such that, for arbitrary multiindices β, $\varphi \in \mathrm{C}^\infty(V_{\mathbb{R}})$, and $g \in G_0$,*

$$(\partial^\beta_m \pi(g)\varphi)(m) = \sum_{|\gamma| \leq |\beta|} b^\gamma_\beta(m,g)\pi(g)d\pi(X^\gamma)\varphi(m), \qquad m \in U. \tag{26}$$

The coefficient functions $b^\gamma_\beta(m,g) \in \mathrm{C}^\infty(U \times G_0)$ are rational expressions in the coordinates of m, and satisfy the estimates

$$|(\partial^\alpha_m b^\gamma_\beta)(m,g)| \leq Ce^{\kappa|g|}, \qquad m \in K \subset\subset U, \tag{27}$$

for some appropriate constants $C, \kappa > 0$ depending on α, β, γ, and K, a compact subset of U.

Proof. By the previous lemma, there exists a neighbourhood U of $\tilde m$ such that, for $m \in U$ and all $g \in G_0$,

$$(\partial_{m_i} \pi(g)\varphi)(m) = \sum_{k=1}^{n} \Theta_i^k(m) d\pi(X_{j_k})\pi(g)\varphi(m).$$

Now,

$$d\pi(X_k)\pi(g)\varphi(m) = \sum_{l=1}^{d} \frac{d}{dh} s_l^k(h,g)_{|_{h=0}} \pi(g) d\pi(X_l)\varphi(m),$$

where the functions s_l^k are given by the equations $\mathrm{e}^{hX_k}\, g = \Phi_g^{-1}(s_1^k(h,g),\dots,s_d^k(h,g)) = g\,\mathrm{e}^{s_1^k(h,g)X_1}\dots \mathrm{e}^{s_d^k(h,g)X_d}$, and are real analytic in h and g [10]. Differentiating these equations with respect to h one obtains

$$g_{ij}(X_k g) = \sum_{l=1}^{d} \frac{\partial}{\partial \zeta_l} (g_{ij} \circ \Phi_g^{-1})(\zeta)_{|_{\zeta=0}} \frac{d}{dh} s_l^k(h,g)_{|_{h=0}}.$$

Let $\chi_{IJ} \equiv \chi_{i_1 j_1,\dots,i_d j_d} \in \mathrm{C}^\infty(G_0)$ be a finite, not necessarily compactly supported, partition of unity subordinate to the covering of G_0 by the sets $G_0^{IJ} \equiv G_0^{i_1 j_1,\dots,i_d j_d}$. By definition, $\partial(g_{i_1 j_1} \circ \Phi_g^{-1},\dots,g_{i_d j_d} \circ \Phi_g^{-1})/\,\partial(\zeta_1,\dots,\zeta_d) \neq 0$ for $g \in G_0^{i_1 j_1,\dots,i_d j_d}$. Since $\frac{d}{d\zeta_l}(g_{ij} \circ \Phi_g^{-1})(\zeta)_{|\zeta=0} = g_{ij}(gX_l)$, we obtain, as a consequence of the last equation,

$$\begin{pmatrix} \dot s_1^k(0,g) \\ \vdots \\ \dot s_d^k(0,g) \end{pmatrix} = \sum_{IJ} \chi_{IJ}(g) \begin{pmatrix} g_{i_1 j_1}(gX_1)\dots & g_{i_1 j_1}(gX_d) \\ \vdots & \vdots \\ g_{i_d j_d}(gX_1)\dots & g_{i_d j_d}(gX_d) \end{pmatrix}^{-1} \begin{pmatrix} g_{i_1 j_1}(X_k g) \\ \vdots \\ g_{i_d j_d}(X_k g) \end{pmatrix}, \tag{28}$$

where $g \in G_0$. The coefficients of g and the functions χ_{IJ} are, together with their derivatives, at most of exponential growth, so that equation (28) implies that the derivatives of the functions $s_l^k(h,g)$ with respect to h satisfy the estimates

$$\left| \frac{d}{dh} s_l^k(h,g)_{|_{h=0}} \right| \leq C e^{\kappa |g|} \tag{29}$$

for some $C > 0$, and $\kappa \geq 1$. Thus we obtain the assertion of the proposition for $|\beta| = 1$ with $b_i^l(m,g) = \sum_{k=1}^n \Theta_i^k(m) \dot s_l^{j_k}(0,g)$. Assume that the assertion holds for $|\beta| \leq N$. The general statement then follows by repeated differentiation. ☐

Corollary 1. *Assume $\tilde m \in V_\mathbb{R} - S_\mathbb{R}$, and let U be a neighbourhood of $\tilde m$ as in the preceeding proposition. Then, for arbitrary multiindices α, $N \in \mathbb{N}$, and $g \in G_0$, the relations*

$$(\partial_\xi^\alpha \pi(g)\varphi_\xi)(m) = \frac{1}{(1+|\xi|^2)^N} \sum_{|\gamma| \leq 2N} d_{\alpha,N}^\gamma(m,g)\pi(g) d\pi(X^\gamma)\varphi_\xi(m), \qquad m \in U, \tag{30}$$

hold, where the coefficients $d_{\alpha,N}^\gamma(m,g) \in \mathrm{C}^\infty(U \times G_0)$ are rational functions in the coordinates of m satisfying the estimates

$$|(\partial_m^\beta d_{\alpha,N}^\gamma)(m,g)| \leq C e^{\kappa|g|}, \qquad m \in K \subset\subset U, \tag{31}$$

for some $C, \kappa > 0$ depending on α, β, γ, N, and K.

Proof. The key step in proving the corollary will be to express $(1+\xi^2)^N$ as a linear combination of derivatives of $\pi(g)\varphi_\xi(m)$ with respect to m. Since $\partial_m^\beta \varphi_\xi(m) = i^{|\beta|}\xi^\beta \varphi_\xi(m)$, one computes

$$(\partial_{m_j} \pi(g)\varphi_\xi)(m) = \sum_{k=1}^n \frac{\partial (g^{-1}m)_k}{\partial m_j} (\partial_{m_k} \varphi_\xi)(g^{-1}m) = i\varphi_\xi(g^{-1}m) \sum_{k=1}^n (g^{-1})_{kj}\xi_k, \tag{32}$$

and repeated differentiation leads to

$$\begin{aligned}(\partial_m^\beta \pi(g)\varphi_\xi)(m) &= i^{|\beta|}\varphi_\xi(g^{-1}m)\big(({}^Tg^{-1})\xi\big)^\beta \\ &= \varphi_{(1-|\beta|)\xi}(g^{-1}m)\cdot \big((\partial_{m_1}\pi(g)\varphi_\xi)(m)\big)^{\beta_1}\cdots\big((\partial_{m_n}\pi(g)\varphi_\xi)(m)\big)^{\beta_n}.\end{aligned} \tag{33}$$

By first taking into account that (32) implies

$$\xi_j = -i\varphi_{-\xi}(g^{-1}m)\sum_{k=1}^n g_{kj}(\partial_{m_k}\pi(g)\varphi_\xi)(m),$$

and then applying (33), we obtain for $(1+\xi^2)^N$ the expression

$$(1+|\xi|^2)^N = \sum_{k=0}^N \binom{N}{k} (\xi_1^2+\cdots+\xi_n^2)^k = \varphi_{-\xi}(g^{-1}m)\sum_{|\beta|\leq 2N} c_\beta(g)(\partial_m^\beta \pi(g)\varphi_\xi)(m),$$

where the $c_\beta(g)$ are rational functions in the matrix coefficients of g. The corollary now follows, since by the previous proposition, one computes for $m\in U$

$$\begin{aligned}(\partial_\xi^\alpha \pi(g)\varphi_\xi)(m) &= (g^{-1}m)^\alpha i^{|\alpha|}\varphi_\xi(g^{-1}m)\\ &= \frac{1}{(1+|\xi|^2)^N}(g^{-1}m)^\alpha i^{|\alpha|}\sum_{|\beta|\leq 2N} c_\beta(g)(\partial_m^\beta\pi(g)\varphi_\xi)(m)\\ &= \frac{1}{(1+|\xi|^2)^N}(g^{-1}m)^\alpha i^{|\alpha|}\sum_{|\beta|\leq 2N} c_\beta(g)\sum_{|\gamma|\leq|\beta|} b_\beta^\gamma(m,g)\pi(g)d\pi(X^\gamma)\varphi_\xi(m),\end{aligned}$$

where the functions $b_\beta^\gamma(m,g)$ are rational expressions in the coordinate functions of m satisfying the bounds (27). □

We are now in position to prove the Structure Theorem.

Proof of Theorem 2. Fix $\tilde m \in V_\mathbb{R} - S_\mathbb{R}$, and assume that $U\subset V_\mathbb{R}-S_\mathbb{R}$ is a neighbourhood of $\tilde m$ as in Proposition 2. Then, by the same proposition, and its corollary, one has

$$\begin{aligned}(\partial_\xi^\alpha\partial_m^\beta \hat f)(m,\xi) &= \int_{G_0} f(g)(\partial_\xi^\alpha\partial_m^\beta\pi(g)\varphi_\xi)(m)d_{G_0}(g)\\ &= \frac{1}{(1+|\xi|^2)^N}\int_{G_0} f(g)\,\partial_m^\beta \sum_{|\gamma|\leq 2N} d_{\alpha,N}^\gamma(m,g)\pi(g)d\pi(X^\gamma)\varphi_\xi(m)d_{G_0}(g)\\ &= \frac{1}{(1+|\xi|^2)^N}\int_{G_0} f(g)\sum_{|\gamma|\leq 2N}\sum_{\delta_1+\delta_2=\beta}\frac{\beta!}{\delta_1!\delta_2!}(\partial_m^{\delta_1} d_{\alpha,N}^\gamma)(m,g)\\ &\cdot\sum_{|\varepsilon|\leq|\delta_2|} b_{\delta_2}^\varepsilon(m,g)\pi(g)d\pi(X^\varepsilon X^\gamma)\varphi_\xi(m)d_{G_0}(g), \qquad m\in U.\end{aligned}$$

Next, for arbitrary $\varphi\in \mathrm{C}^\infty(V_\mathbb{R})$ and $X\in\mathfrak{U}$,

$$\pi(g)d\pi(X)\varphi(m) = dR(X)\varphi_m(g), \qquad g\in G_0,\, m\in V_\mathbb{R}, \tag{34}$$

where we set $\varphi_m(g) = \pi(g)\varphi(m)$. Indeed,

$$\pi(g)d\pi(X_i)\varphi(m) = \lim_{h\to 0} h^{-1}[\varphi(\mathrm{e}^{-hX_i} g^{-1}m) - \varphi(g^{-1}m)] = \lim_{h\to 0} h^{-1}[\varphi_m(g\,\mathrm{e}^{hX_i}) - \varphi_m(g)],$$

so (34) is correct for $X \in \mathfrak{g}$. Assuming that the assertion holds for $X^\gamma \in \mathfrak{U}$ with $|\gamma| = N$, we obtain for arbitrary X_i

$$\begin{aligned}\pi(g)d\pi(X_iX^\gamma)\varphi(m) &= \lim_{h\to 0} h^{-1}[d\pi(X^\gamma)\varphi(\mathrm{e}^{-hX_i} g^{-1}m) - d\pi(X^\gamma)\varphi(g^{-1}m)]\\ &= \lim_{h\to 0} h^{-1}[dR(X^\gamma)\varphi_m(g\,\mathrm{e}^{hX_i}) - dR(X^\gamma)\varphi_m(g)] = dR(X_iX^\gamma)\varphi_m(g),\end{aligned}$$

so that, by induction, we get (34) for arbitrary $X \in \mathfrak{U}$. Now, integrating by parts according to (7) yields

$$\begin{aligned}(\partial_\xi^\alpha \partial_m^\beta \hat{f})(m,\xi) &= \frac{1}{(1+|\xi|^2)^N} \sum_{|\gamma|\leq 2N} \sum_{\delta_1+\delta_2=\beta} \sum_{|\varepsilon|\leq|\delta_2|} \int_{G_0} F^{\alpha,\beta,N}_{\delta_1,\delta_2,\gamma,\varepsilon}(m,g) dR(X^\varepsilon X^\gamma)\varphi_{\xi,m}(g) d_{G_0}(g)\\ &= \frac{1}{(1+|\xi|^2)^N} \sum_{|\gamma|\leq 2N} \sum_{\delta_1+\delta_2=\beta} \sum_{|\varepsilon|\leq|\delta_2|} \sum_{\sigma_1+\sigma_2=\varrho(\varepsilon,\gamma)} \iota^{\sigma_1}(-1)^{|\sigma_2|}\\ &\cdot \int_{G_0} dR(X^{\tilde{\sigma}_2}) F^{\alpha,\beta,N}_{\delta_1,\delta_2,\gamma,\varepsilon}(m,g)\pi(g)\varphi_\xi(m) d_{G_0}(g),\end{aligned}$$

where we set

$$F^{\alpha,\beta,N}_{\delta_1,\delta_2,\gamma,\varepsilon}(m,g) = \frac{\beta!}{\delta_1!\delta_2!} f(g)(\partial_m^{\delta_1} d^\gamma_{\alpha,N})(m,g) b^\varepsilon_{\delta_2}(m,g),$$

$\varphi_{\xi,m}(g) = \pi(g)\varphi_\xi(m)$, and $X^{\varrho(\varepsilon,\gamma)} = X^\varepsilon X^\gamma$. Note that, for fixed $m \in U$, $F^{\alpha,\beta,N}_{\delta_1,\delta_2,\gamma,\varepsilon}(m,g) \in \mathcal{S}(G_0)$, as a consequence of the estimates (27) and (31), so that integration by parts is legitimate. Let ω be a compact set in $V_\mathbb{R} - S_\mathbb{R}$. For each point $\tilde{m} \in \omega$, let $U_{\tilde{m}} \subset V_\mathbb{R} - S_\mathbb{R}$ be a neighbourhood of $\tilde{m}$ as in proposition 2. By the Theorem of Heine-Borel, ω can be covered by finitely many $U_{\tilde{m}}$, so that $\bigcup_{\tilde{m}\in\omega} U_{\tilde{m}}$ has a finite subcovering. Fix $l \in \mathbb{R}$, and let k be an integer $\leq l$. Assume that α, β are arbitrary multiindices. Setting $N = |\alpha| + |k|$, and taking into account the above expression for $(\partial_\xi^\alpha \partial_m^\beta \hat{f})(m,\xi)$, one computes

$$|\partial_\xi^\alpha \partial_m^\beta \hat{f}(m,\xi)| \leq C_{\alpha,\beta,N,\omega} \langle\xi\rangle^{-2N} \leq C_{\alpha,\beta,k,\omega} \langle\xi\rangle^{k-|\alpha|} \leq C_{\alpha,\beta,l,\omega} \langle\xi\rangle^{l-|\alpha|}, \qquad m \in \omega,$$

for some suitable constants. Thus, $\hat{f}(m,\xi) \in \mathrm{S}^l((V_\mathbb{R} - S_\mathbb{R}) \times V^*_\mathbb{R})$ for all $l \in \mathbb{R}$, yielding $a_f(m,\xi) \in \mathrm{S}^{-\infty}((V_\mathbb{R} - S_\mathbb{R}) \times V^*_\mathbb{R})$. Now, for $\varphi \in \mathrm{C}^\infty_\mathrm{c}(V_\mathbb{R} - S_\mathbb{R})$, Fourier transformation gives

$$\begin{aligned}\pi(f)\varphi(m) &= \int_{G_0} f(g)\pi(g)\varphi(m)\, d_{G_0}(g) = \int_{G_0} f(g) \left(\int e^{i(g^{-1}m)\cdot\xi} \hat{\varphi}(\xi) đ\xi \right) d_{G_0}(g)\\ &= \int \hat{f}(m,\xi)\hat{\varphi}(\xi) đ(\xi) = \int\int e^{i(m-m')\cdot\xi} a_f(m,\xi)\varphi(m')\, dm'\, đ\xi,\end{aligned}$$

where the occurring integrals are absolutely convergent, so that we can interchange the order of integration. This proves Theorem 2. □

5. Fixed points

Let (G, ϱ, V) be a prehomogeneous vector space with singular set S. In this section, we restrict our attention to the case where $S_\mathbb{R} = S \cap V_\mathbb{R}$ coincides with the set of fixed points $F_\mathbb{R}$ of the $G_\mathbb{R}$-action. If $G_\mathbb{R}$ is reductive, every fiber of the quotient morphism $\sigma : V_\mathbb{R} \to V_\mathbb{R}/G_\mathbb{R}$ contains exactly one closed orbit (see [9], page 189), so that zero constitutes the only closed orbit contained in $S_\mathbb{R}$. Therefore, if $S_\mathbb{R} = F_\mathbb{R}$, we necessarily must have $S_\mathbb{R} = \{0\}$. This will be the case we will be mainly concerned with. Note that if a reductive group G acts on a smooth affine variety M, and if M contains exactly one fixed point, then M is a vector space on which G acts linearly (see [9], page 214). If $m \in V_{\mathbb{R}}$ is a fixed point, the symbol of $\pi(f)$ at m is simply given by $a_f(m,\xi) = \int_{G_0} f(g)\, d_{G_0}(g)$. Thus, if $S_{\mathbb{R}} = F_{\mathbb{R}}$, one has

$$a_f(m,\xi) \in \left\{ \begin{array}{ll} \mathrm{S}^{-\infty}((V_{\mathbb{R}})_{m'} \times V_{\mathbb{R}}^*) & \text{if } m \in V_{\mathbb{R}} - F_{\mathbb{R}}, \\ \mathrm{S}^{0}((V_{\mathbb{R}})_{m'} \times V_{\mathbb{R}}^*) & \text{if } m \in F_{\mathbb{R}}. \end{array} \right. \tag{35}$$

In this situation, Theorem 2 can be restated as follows.

Theorem 3. *Let (G, ϱ, V) be a prehomogeneous vector space defined over $\mathbb{R}$, and assume that $S_{\mathbb{R}}$ is equal to the set $F_{\mathbb{R}}$ of fixed points of the underlying $G_{\mathbb{R}}$-action. Then $\pi(f) : \mathrm{C}_{\mathrm{c}}^{\infty}(V_{\mathbb{R}}) \to \mathcal{D}'(V_{\mathbb{R}})$ is given by the family of oscillatory integrals*

$$(\pi(f)\varphi)(m) = \int\int e^{i(m-m')\cdot\xi} a_f(m,\xi)\varphi(m')dm' d\xi,$$

where

$$a_f(m,\xi) = e^{-im\cdot\xi}\hat{f}(m,\xi) \in \mathrm{S}^0((V_{\mathbb{R}})_{m'} \times V_{\mathbb{R}}^*), \qquad m \in V_{\mathbb{R}};$$

for $m \notin F_{\mathbb{R}}$, one has $a_f(m,\xi) \in \mathrm{S}^{-\infty}((V_{\mathbb{R}})_{m'} \times V_{\mathbb{R}}^*)$. Its Schwartz kernel $\mathcal{K}_f \in \mathcal{D}'(V_{\mathbb{R}} \times V_{\mathbb{R}})$ is given by the continuous family of oscillatory integrals

$$V_{\mathbb{R}} \ni m \mapsto \mathcal{K}_{f,m} = \int e^{i(m-\cdot)\cdot\xi} a_f(m,\xi) d\xi.$$

Furthermore, one has $a_f(m,\xi) \in \mathrm{S}^{-\infty}((V_{\mathbb{R}} - F_{\mathbb{R}})_m \times (V_{\mathbb{R}})_{m'} \times V_{\mathbb{R}}^*)$ so that, in particular, $\pi(f) \in \mathrm{L}^{-\infty}(V_{\mathbb{R}} - F_{\mathbb{R}})$.

Example 1. Consider the simplest prehomogeneous vector space, $(\mathbb{C}^*, \mathbb{C})$, where $\mathbb{C}^*$ acts by multiplication on $\mathbb{C}$. Then $S = S_{\mathbb{R}} = \{0\}$, and $p(m) = m$ is a relative invariant corresponding to $\chi(g) = g$, $g \in \mathbb{C}^*$. One has $V_{\mathbb{R}} - S_{\mathbb{R}} = \mathbb{R}^* = V_1 \cup V_2$ with $V_1 = \mathbb{R}^*_+$, $V_2 = \mathbb{R}^*_-$. The Fundamental Theorem yields in this case the relations

$$\int_{-\infty}^{\infty} |\xi|^{s-1}\hat{\varphi}(\xi)\, d\xi = (2\pi)^{-s}\Gamma(s)\, 2\cos\frac{\pi s}{2} \int_{-\infty}^{\infty} |m|^{-s}\varphi(m)\, dm, \qquad \varphi \in \mathcal{S}(\mathbb{R}), \tag{36}$$

(see [5], Proposition 4.21).

Example 2. Let B be a positive definite real symmetric matrix of degree $n \geq 3$, and consider the positive definite quadratic forms $p(m) = m^tBm$, $p^*(m) = m^tB^{-1}m$ on $\mathbb{C}^n$. We define $\mathrm{SO}(n,B) = \{X \in \mathrm{SL}(n,\mathbb{C}) : X^tBX = B\}$. Setting $\varrho(\alpha, g)m = \alpha g m$, $\varrho^*(\alpha, g) = \alpha^{-1}(g^{-1})^t m$, where $\alpha \in \mathbb{C}^*$, $g \in \mathrm{SO}(n,B)$, one obtains actions of $G = \mathrm{GL}(1,\mathbb{C}) \times \mathrm{SO}(n,B)$ on $V = V^* = \mathbb{C}^n$. Again, $S_{\mathbb{R}} = \{0\}$, and by the Fundamental Theorem one obtains the functional equations

$$\int_{\mathbb{R}^n - \{0\}} |p^*(\xi)|^{s-\frac{n}{2}}\hat{\varphi}(\xi)\, d\xi = \pi^{\frac{n}{2}-s}\sqrt{\det B}\frac{\Gamma(s)}{\Gamma(\frac{n}{2}-s)} \int_{\mathbb{R}^n - \{0\}} |p(m)|^{-s}\varphi(m)\, dm, \tag{37}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ (see [5], Proposition 4.22).

Returning to the situation of the previous theorem, note that on $(V_{\mathbb{R}} - F_{\mathbb{R}}) \times V_{\mathbb{R}}$, the Schwartz kernel $\mathcal{K}_f$ of $\pi(f)$ is given by the function

$$K_f(m,m') = \int e^{i(m-m')\cdot\xi} a_f(m,\xi) d\xi \in \mathrm{C}^{\infty}((V_{\mathbb{R}} - F_{\mathbb{R}}) \times V_{\mathbb{R}}),$$

while, as distributions, $\mathcal{K}_{f,m} \to \mathcal{K}_{f,x_F} = \int_{G_0} f(g) d_{G_0}(g) \cdot \delta_{m_F}$ as $m \to m_F \in F_{\mathbb{R}}$. In this way, the orbit structure of the underlying $G_{\mathbb{R}}$-action is reflected in the singular behaviour of the Schwartz kernel of $\pi(f) : \mathrm{C}_{\mathrm{c}}^{\infty}(V_{\mathbb{R}}) \to \mathcal{D}'(V_{\mathbb{R}})$ at $F_{\mathbb{R}}$. In what follows, we will assume that $S_{\mathbb{R}} = \{0\}$. In order to get a better understanding of the Schwartz kernel $\mathcal{K}_f$ of $\pi(f)$, and, in particular, of its restriction to the diagonal, we define the auxiliary symbol

$$\tilde{a}_f(m,\xi) = e^{-i\frac{m\cdot\xi}{|m|}} \hat{f}(m/|m|,\xi) = a_f(m/|m|,\xi), \qquad m \neq 0.$$

By introducing the inverse Fourier transform

$$\tilde{A}_f(m,m') = \int e^{im'\cdot\xi} \tilde{a}_f(m,\xi) đ\xi$$

of $\tilde{a}(m,\xi)$, and since $a_f(\alpha m,\xi) = a_f(m,\alpha\xi)$, $\alpha \in \mathbb{R}$, we obtain the relation

$$K_f(m,m') = \int e^{i(m-m')\cdot\xi} \tilde{a}_f(m,|m|\xi) đ\xi = \frac{1}{|m|^n} \tilde{A}_f\Big(m, \frac{m-m'}{|m|}\Big), \quad m \neq 0.$$

Lemma 2. $K_f(m,m') \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{2n}) \cap \mathrm{C}^\infty(\mathbb{R}^{2n} - \{0\})$.

Proof. By the Structure Theorem, $\tilde{a}_f(m,\xi) \in \mathrm{S}^{-\infty}(V_{\mathbb{R}} - \{0\}, V_{\mathbb{R}})$, which implies that $\tilde{A}_f(m,m')$ vanishes to all orders as $|m'| \to \infty$. Since $\tilde{A}_f(m,m')$ only depends on the direction of m, $\tilde{A}_f(m,(m-m')/|m|)$ goes to zero to infinite order as $|m| \to 0$, provided $m' \neq 0$. By setting $K_f(m,m') = 0$ for $m = 0$, $m' \neq 0$, we therefore obtain $K_f(m,m') \in \mathrm{C}^\infty(\mathbb{R}^{2n} - \{0\})$. Let L be a compact set in $\mathbb{R}^{2n}$. Then, for arbitrary N,

$$\left| \int_L K_f(m,m')\, dm\, dm' \right| \leq C_N \int_{L-\{0\}} \frac{|m|^{2N-n}}{(|m|^2 + |m-m'|^2)^N}\, dm\, dm'.$$

For $N = n/2$, the last integral reads

$$\int_{L-\{0\}} \frac{dm\, dm'}{(|m|^2 + |m-m'|^2)^{n/2}} = \int_0^R \int_{S^{2n-1}} \frac{r^{2n-1}\, dr\, d\omega}{r^n \sum_{i=1}^n \omega_i^2 + (\omega_i - \omega_{n+i})^2},$$

where we introduced in $V_{\mathbb{R}} \times V_{\mathbb{R}} \simeq \mathbb{R}^{2n}$ the polar coordinates $r \in \mathbb{R}^+$ and $\omega \in S^{2n-1} \subset \mathbb{R}^{2n}$, and assumed that L is contained in a sphere of radius R. Since $\sum_{i=1}^n \omega_i^2 + (\omega_i - \omega_{n+i})^2 > 0$ is a real valued, continuous function on the $(2n-1)$-dimensional sphere which is bounded from below, the infimum is adopted. Hence, there is a strictly positive number κ such that $\sum_{i=1}^n \omega_i^2 + (\omega_i - \omega_{n+i})^2 \geq \kappa$. This proves the lemma. □

The last lemma implies that $\mathcal{K}_f$ is given by the locally integrable function $K_f(m,m')$. Let us examine its restriction to the diagonal. By the Structure Theorem,

$$k_f(m) = K_f(m,m) = \int a_f(m,\xi) đ\xi = \frac{1}{|m|^n} \tilde{A}_f(m,0) \in \mathrm{C}^\infty(V_{\mathbb{R}} - \{0\}).$$

We then have the following proposition.

Proposition 3. *Let (G,ϱ,V) be a reductive prehomogeneous vector space, and $S_{\mathbb{R}} = \{0\}$. Set $k_{f,s}(m) = |m|^s \tilde{A}_f(m,0)$. Then, for $\varphi \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$, the integrals*

$$\int_{\mathbb{R}^n - \{0\}} k_{f,s}(m)\varphi(m)\, dm,$$

can be continued to meromorphic functions on the whole s-plane, with simple poles at $s = -n, -(n+1), \ldots$. For $\varphi \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n - \{0\})$, they satisfy the functional equations

$$\int_{\mathbb{R}^n - \{0\}} |\xi|^{2(s-\frac{n}{2})} \widehat{\tilde{A}_f(\cdot,0)} \varphi(\xi)\, d\xi = \pi^{\frac{n}{2}-2s} \frac{\Gamma(s)}{\Gamma\Big(\frac{n}{2} - s\Big)} \int_{\mathbb{R}^n - \{0\}} k_{f,-2s}(m)\varphi(m)\, dm.$$

Proof. Put $p(m) = m_1^2 + \dots + m_n^2 = |m|^2$, and assume $\mathrm{Re}\, s > -n/2$. One then computes

$$\int_{\mathbb{R}^n - \{0\}} k_{f,s}(m)\varphi(m)\, dm = \int_{\mathbb{R}^n - \{0\}} |p(m)|^{s/2} \tilde{A}_f(m,0)\varphi(m)\, dm$$
$$= \int_{S^{n-1}} \tilde{A}_f(\omega,0) \int_0^\infty \varphi(r\omega) r^{s+n-1} dr d\omega,$$

since everything in sight is absolutely convergent. Consider now the n-dimensional Mellin transform

$$M_s : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathrm{C}^\infty(S^{n-1}), \qquad M_s(\varphi)(\omega) = \int_0^\infty \varphi(r\omega) r^s dr, \qquad \mathrm{Re}\, s > -1,$$

which can be continued meromorphically in s with simple poles at $s = -1, -2, \dots$. Since, by the above computation,

$$\int_{\mathbb{R}^n - \{0\}} k_{f,s}(m)\varphi(m)\, dm = \int_{S^{n-1}} \tilde{A}_f(\omega,0) M_{s+n-1}(\varphi)(\omega) d\omega,$$

we obtain the first assertion. The second one follows directly from the relations (37). □

Now, although the integrals $\int_{\mathbb{R}^n - \{0\}} k_f(m)\varphi(m)\, dm$ become singular for general $\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^n)$, it is nevertheless possible to extend k_f, as a distribution, to $\mathbb{R}^n$. First note that, since $K_f(tm,tm) = t^{-n} K_f(m,m)$ on $V_\mathbb{R} - \{0\}$, where $t > 0$, $k_f(m)$ defines a homogeneous distribution on $V_\mathbb{R} - \{0\}$ of degree $-n$ according to

$$\langle k_f, \varphi \rangle = \int_{|\omega|=1} \int_0^\infty k_f(w) r^{-1} \varphi(r\omega)\, dr\, d\omega.$$

By [3], Theorem 3.2.4, an extension of k_f to $V_\mathbb{R}$ can then be constructed as follows. Let s be a complex number with $\mathrm{Re}\, s > -1$, and consider on $\mathbb{R}$ the locally integrable function

$$t_+^s = t^s, \quad \text{if } t > 0, \qquad t_+^s = 0 \quad \text{if } t \leq 0.$$

It is homogeneous of degree s, and can be extended, as a distribution, to arbitrary complex values of s by analytic continuation, except for simple poles at $s = -1, -2, \dots$. The residue of the function $s \mapsto t_+^s(\varphi)$ at $s = -k$, where $\varphi \in \mathrm{C}^\infty_c(\mathbb{R})$, is given by

$$\lim_{s \to -k} (s+k) t_+^s(\varphi) = \varphi^{(k-1)}(0)/(k-1)!,$$

so that one defines $t_+^{-k}(\varphi)$ as the limit of $t_+^s(\varphi) - \varphi^{(k-1)}(0)/((k-1)!(s+k))$ as $s \to -k$, getting in this way

$$t_+^{-k}(\varphi) = -\frac{1}{(k-1)!} \int_0^\infty \log t \varphi^{(k)}(t) dt + \frac{1}{(k-1)!} \varphi^{(k-1)}(0) \Big(\sum_{j=1}^{k-1} \Big) j^{-1}.$$

For $s \neq -1, -2, \dots$, t_+^s is a homogeneous distribution on $\mathbb{R}$. Now, for $m \neq 0$, and $\varphi \in \mathrm{C}^\infty_c(V_\mathbb{R})$, set $R_s \varphi(m) = t_+^{s+n-1}(\varphi(tm))$, and let $\psi \in \mathrm{C}^\infty_c(V_\mathbb{R} - \{0\})$ satisfy

$$\int_0^\infty \psi(tm) \frac{dt}{t} = 1, \quad m \neq 0.$$

Then, the extension of k_f is defined as

$$\dot{k}_f(\varphi) = \langle k_f, \psi R_{-n} \varphi \rangle, \qquad \varphi \in \mathrm{C}^\infty_c(V_\mathbb{R}). \tag{38}$$

Summing up, we have obtained the following proposition.

Proposition 4. *Let (G, ϱ, V) be a prehomogeneous vector space, and $S_{\mathbb{R}} = \{0\}$. Then, the restriction of the kernel of $\pi(f) \in \mathrm{L}^{-\infty}(V_{\mathbb{R}} - \{0\})$ to the diagonal is given by the homogeneous distribution $k_f = K_f(m, m) \in \mathrm{C}^{\infty}(V_{\mathbb{R}} - \{0\})$ of degree $-n$, which has an extension to $V_{\mathbb{R}}$ given by* (38).

Remark 2. The regularization procedure employed above is essentially due to Riesz. The extension $\dot{k}_f$ is unique up to a linear combination of δ-distributions at zero, and is no longer homogeneous. One could also have defined $\dot{k}_f$ by

$$\dot{k}_f(\varphi) = \int_{|\omega|=1} k_f(\omega) R_{-n}\varphi(\omega)\, d\omega, \qquad \varphi \in \mathrm{C}^{\infty}_{\mathrm{c}}(V_{\mathbb{R}}).$$

Each of the extensions $\dot{k}_f$, which are obtained by subtracting the singular part of $k_{f,s}(\varphi)$ at $-n$, could then be regarded as a trace of $\pi(f)$. Note that the residue of $k_{f,s}(\varphi)$ at $s = -n$ is $\varphi(0) \int_{|\omega|=1} \tilde{A}_f(\omega, 0)\, d\omega$.

6. Totally characteristic pseudodifferential operators on prehomogeneous vector spaces

Let (G, ϱ, V) be a reductive prehomogeneous vector space defined over $\mathbb{R}$, and assume that the singular set S is an irreducible hypersurface. Denote by p the corresponding relative invariant with character χ such that (10) is satisfied, and by $(\pi, \mathrm{C}_0(V_{\mathbb{R}}))$ the left regular representation of $G_{\mathbb{R}}$ on the Banach space $\mathrm{C}_0(V_{\mathbb{R}})$. As already explained in Section 4, given a function $f \in \mathcal{S}(G_0)$, the restriction of the operator $\pi(f)$ to $\mathrm{C}^{\infty}_{\mathrm{c}}(V_{\mathbb{R}})$ defines a continuous linear operator

$$\pi(f) : \mathrm{C}^{\infty}_{\mathrm{c}}(V_{\mathbb{R}}) \to \mathrm{C}^{\infty}(V_{\mathbb{R}}).$$

Now, if $V_{\mathbb{R}} - S_{\mathbb{R}} = V_1 \cup \dots \cup V_l$ denotes the decomposition of the generic $G_{\mathbb{R}}$-orbit into its connected components, the restriction of $\pi(f)\varphi$ to V_i only depends on the restriction of $\varphi \in \mathrm{C}_0(V_{\mathbb{R}})$ to V_i, so that one naturally obtains the continuous linear operators

$$\pi(f)_{|V_i} : \mathrm{C}^{\infty}_{\mathrm{c}}(V_i) \longrightarrow \mathrm{C}^{\infty}(V_i),$$

which are elements in $\mathrm{L}^{-\infty}(V_i)$, by the Structure Theorem. Let $S_{\mathbb{R}}^{\mathrm{sing}}$, respectively $(\partial V_i)^{\mathrm{sing}}$, be the set of non-regular points of $S_{\mathbb{R}}$, respectively ∂V_i. The main goal of this section is to prove Theorem 4, which gives a precise description of the kernel of $\pi(f)$ as an operator on $V_{\mathbb{R}} - S_{\mathbb{R}}^{\mathrm{sing}}$, under certain assumptions regarding the $G_{\mathbb{R}}$-action of the underlying prehomogeneous vector space. As an immediate consequence, Proposition 5 states that the restrictions of the operators $\pi(f)$ to $\overline{V}_i - (\partial V_i)^{\mathrm{sing}}$ are totally characteristic pseudodifferential operators of class $\mathrm{L}_b^{-\infty}$, with the singular set $S_{\mathbb{R}}$ playing the role of the boundary.

Remark 3. If (G, ϱ, V) is a reductive prehomogeneous vector space whose singular set is an irreducible hypersurface, the Hessian of its relative invariant p is non-singular on $V - S$. Let us assume that it is also non-singular on S, as in Example 3 below. Then $S_{\mathbb{R}}^{\mathrm{sing}} = \{0\}$. Indeed, as already explained at the beginning of section 5, zero is the only closed orbit in $S_{\mathbb{R}}$, and by the Lemma of Morse (see e.g. [8], page 8), every non-degenerate critical point must be isolated; this implies that zero is the only critical point of p in $S_{\mathbb{R}}$.

In order to formulate our assumptions, we define the closed subgroups

$$G(p) = \{g \in G : p(gm) = p(m)\} = \{g \in G : \chi(g) = 1\}, \qquad G_{\mathbb{R}}(p) = G(p) \cap G_{\mathbb{R}}.$$

Clearly, $G_{\mathbb{R}}(p)$ acts transitively on each generic fiber of the categorical quotient $\sigma : V_{\mathbb{R}} \to V_{\mathbb{R}}/G_{\mathbb{R}}$, and its Lie algebra is given by $\mathfrak{g}(p) = \{X \in \mathfrak{g} : d\chi(X) = 0\}$, where $d\chi$ denotes the infinitesimal representation corresponding to χ. There always exists a $Y \in \mathfrak{g}$ such that

$d\chi(Y) \neq 0$; otherwise, $V_\mathbb{R} - S_\mathbb{R}$ would decompose into an infinite number of G_0-orbits. Our assumptions then read as follows.

Assumption 1. *$S_\mathbb{R} = S \cap V_\mathbb{R}$ is an irreducible hypersurface.*

Assumption 2. *For each $m \in V_\mathbb{R} - S_\mathbb{R}^{\text{sing}}$, there exists an open neighbourhood $Z \subset V_\mathbb{R} - S_\mathbb{R}^{\text{sing}}$ of m and elements $X_1, \dots, X_{n-1} \in \mathfrak{g}(p)$ such that $m' \to (\tilde{X}_1(m'), \dots, \tilde{X}_{n-1}(m'))$ defines a section in $T(p^{-1}(c) \cap Z)$ for each $c \in \mathbb{R}$.*

Here we wrote $\tilde{X}(m) = \frac{d}{dh}(\varrho(\,e^{hX}\,)m)_{|h=0}$ for the fundamental vector field of the underlying $G_\mathbb{R}$-action on $V_\mathbb{R}$ induced by $X \in \mathfrak{g}$. The second assumption basically states that $G_\mathbb{R}(p)$ acts locally transitively on $S_\mathbb{R}^{\text{reg}} = S_\mathbb{R} - S_\mathbb{R}^{\text{sing}}$. Note that, locally, the vector fields $\tilde{Y}, \tilde{X}_1, \dots, \tilde{X}_{n-1}$ generate the algebra of totally characteristic differential operators $\text{Diff}_\text{b}(\overline{\text{V}}_\text{i} - (\partial\,\text{V}_\text{i})^{\text{sing}})$ (see [4], page 113). Let us illustrate these assumptions by two examples.

Example 3. As the generic case in the theory of prehomogeneous vector spaces, consider an indefinite quadratic form p with signature $(q, n-q)$, $1 \leq q \leq n-1$, on $\mathbb{R}$. Then, by Sylvester law, $p(m) = B^t \mathbf{1}_{q,n-q} m^t B m$, for some $B \in \text{GL}(n, \mathbb{R})$. Define the orthogonal, respectively special orthogonal, group of p by $\text{O}(p) = \{g \in \text{GL}(n, \mathbb{C}) : p(gm) = p(m)\}$, respectively $\text{SO}(p) = \{g \in \text{SL}(n, \mathbb{C}) : p(gm) = p(m)\}$, and introduce on $V = \mathbb{C}^n$ an action of $G = \text{GL}(1, \mathbb{C}) \times \text{SO}(p)$ by setting $\varrho(\alpha, g)m = \alpha g m$. Then p is an irreducible relative invariant of (G, ϱ, V), and $V_\mathbb{R} - S_\mathbb{R} = V_+ \cup V_-$, where $V_\pm = \{m \in V_R : \pm p(m) > 0\}$. By the Theorem of Witt, $p^{-1}(0) - \{0\}$ is $\text{O}(p)$-homogeneous, and $\text{SO}(p)$-homogeneous for $n \geq 3$. Hence, the Assumptions 1 and 2 are clearly satisfied. In this case, the Fundamental Theorem yields the functional equations

$$\begin{pmatrix} \int_{V_+^*} |p^*(\xi)|^{s-\frac{n}{2}} \hat{\varphi}(\xi)\, d\xi \\ \int_{V_-^*} |p^*(\xi)|^{s-\frac{n}{2}} \hat{\varphi}(\xi)\, d\xi \end{pmatrix} = C(s) \begin{pmatrix} \int_{V_+} |p(m)|^{-s} \varphi(m)\, dm \\ \int_{V_-} |p(m)|^{-s} \varphi(m)\, dm \end{pmatrix}, \tag{39}$$

where $\varphi \in \mathcal{S}(V_\mathbb{R})$, and

$$C(s) = \Gamma\Big(s + 1 - \frac{n}{2}\Big)\Gamma(s) |\det B| \pi^{-2s+\frac{n}{2}-1} \begin{pmatrix} -\sin \pi(s - \frac{q}{2}) & \sin \frac{\pi q}{2} \\ \sin \frac{\pi(n-q)}{2} & -\sin \pi(s - \frac{n-q}{2}) \end{pmatrix},$$

see [5], Proposition 4.27.

Example 4. As an example of a non-regular prehomogeneous vector space, consider the subgroup G of upper triangular 2×2-matrices in $\text{GL}(2, \mathbb{C})$ acting regularly on $V = \mathbb{C}^2$. In this case, $S = \big\{m \in \mathbb{C}^2 : m_2 = 0\big\}$. As one verifies, Assumption 2 is not satisfied.

We introduce now local coordinates in $V_\mathbb{R} - S_\mathbb{R}^{\text{sing}}$. Since the fibers of the categorical quotient $p : V_\mathbb{R} \to V_\mathbb{R}/G_\mathbb{R}$ are, apart from the exceptional divisor $S_\mathbb{R}$, smooth affine varieties, we have $\text{grad}\, p(m) \neq 0$ on $V_\mathbb{R} - S_\mathbb{R}^{\text{sing}}$. Defining the open subsets

$$(V_\mathbb{R} - S_\mathbb{R}^{\text{sing}})_j = \Big\{m \in V_\mathbb{R} - S_\mathbb{R}^{\text{sing}} : (\partial\,/\,\partial_{m_j}\, p)(m) \neq 0\Big\},$$

and the coordinates

$$\kappa(m) = (x_1(m), \dots, x_n(m)), \qquad x_1(m) = p(m), \quad x_2(m) = m_{l_1}, \dots, x_n(m) = m_{l_{n-1}}$$

on $(V_\mathbb{R} - S_\mathbb{R}^{\text{sing}})_j$, where $\{l_1, \dots, l_{n-1}\} \cup \{j\} = \{1, \dots, n\}$, one gets

$$\det \left(\frac{\partial\, x_i}{\partial\, m_k}(m)\right)_{ik} \neq 0 \qquad \text{for } m \in (V_\mathbb{R} - S_\mathbb{R}^{\text{sing}})_j. \tag{40}$$

By the Inverse Function Theorem, there exists for every $m \in (V_\mathbb{R} - S_\mathbb{R}^{\text{sing}})_j$ an open neighbourhood $W_j \subset (V_\mathbb{R} - S_\mathbb{R}^{\text{sing}})_j$ of m such that $\kappa : W_j \to \tilde{W}_j = \kappa(W_j) \subset \mathbb{R}^n$ becomes a diffeomorphism, and we obtain a covering of $V_\mathbb{R} - S_\mathbb{R}^{\text{sing}}$ by charts. Fix $m \in (V_\mathbb{R} - S_\mathbb{R}^{\text{sing}})_j$, and let W_j and Z be neighbourhoods of m as specified above. Let U, U_1 be open sets containing m such that $U \subset U_1 \subset W_j \cap Z$, and assume that U is chosen in such a way that the set $\{g \in G_0 : gU \subset U_1\}$ acts transitively on the G_0-orbits of U. The so defined sets U constitute an open covering of $V_\mathbb{R} - S_\mathbb{R}^{\text{sing}}$. By choosing a locally finite subcovering, we therefore obtain an atlas $\{(\kappa_\gamma, U^\gamma)\}$ of $V_\mathbb{R} - S_\mathbb{R}^{\text{sing}}$ with the following properties:

(i) the coordinates of $m \in U^\gamma$ are $\kappa_\gamma(m) = (x_1, \dots, x_n) = (p(m), m_{l_1}, \dots, m_{l_{n-1}})$, where $\{l_1, \dots, l_{n-1}\} \cup \{j\} = \{1, \dots, n\}$ for some $j = j(\gamma)$;
(ii) for each U^γ, there exist open sets $G^\gamma \subset G_1^\gamma \subset G_0$, stable under inverse, acting transitively on the G_0-orbits of U^γ;
(iii) for $m \in \bigcup_{g \in G_1^\gamma} gU^\gamma$, one has $\frac{\partial}{\partial m_j} p(m) \neq 0$ for $j = j(\gamma)$;
(iv) there exist $X_1, \dots, X_{n-1} \in \mathfrak{g}(p)$, such that $m \to (\tilde{X}_1(m), \dots, \tilde{X}_{n-1}(m))$ defines a section in $T(p^{-1}(c) \cap \bigcup_{g \in G_1^\gamma} gU^\gamma)$ for each $c \in \mathbb{R}$.

Consider now with respect to the atlas $\{(\kappa_\gamma, U^\gamma)\}$ the local operators $A_f^\gamma u = [\pi(f)_{|U^\gamma}(u \circ \kappa_\gamma)] \circ \kappa_\gamma^{-1}$ introduced in Section 3, where $u \in \mathrm{C}_c^\infty(\tilde{U}^\gamma)$. They are given explicitely by

$$A_f^\gamma u(x) = \int_{G_0} f(g)\pi(g)(u \circ \kappa_\gamma)(\kappa_\gamma^{-1}(x)) d_{G_0}(g), \qquad x \in \tilde{U}^\gamma.$$

Let $c_\gamma \in \mathrm{C}^\infty(G_0)$ be a smooth function on G_0 with support in G_1^γ satisfying $c_\gamma \equiv 1$ on G^γ. Then A_f^γ can be written as

$$A_f^\gamma u(x) = \int_{G_0} f(g)(u \circ \kappa_g^\gamma)(x) c_\gamma(g) d_{G_0}(g), \qquad x \in \tilde{U}^\gamma,$$

where we put $\kappa_g^\gamma = \kappa_\gamma \circ g^{-1} \circ \kappa_\gamma^{-1}$. We define

$$\hat{f}_\gamma(x, \xi) = \int_{G_0} f(g) e^{i\kappa_g^\gamma(x)\cdot\xi} c_\gamma(g) d_{G_0}(g), \qquad a_f^\gamma(x, \xi) = e^{-ix\cdot\xi} \hat{f}_\gamma(x, \xi). \tag{41}$$

By the Theorem of Lebesgue on Dominated Convergence, we can differentiate under the integral sign, yielding $\hat{f}_\gamma(x, \xi)$, $a_f^\gamma(x, \xi) \in \mathrm{C}^\infty(\tilde{U}^\gamma \times \mathbb{R}_\xi^n)$. Set

$$T_x = \begin{pmatrix} x_1 & 0 \\ 0 & \mathbf{1}_{n-1} \end{pmatrix}, \qquad x \in \tilde{U}^\gamma,$$

and define the symbol

$$\tilde{a}_f^\gamma(x, \xi) = a_f^\gamma(x, T_x^{-1}\xi), \qquad x_1 \neq 0.$$

Note that, by equation (10), $p(g\kappa_\gamma^{-1}(x)) = \chi(g)p(\kappa_\gamma^{-1}(x)) = \chi(g)x_1$, so that

$$T_x^{-1}\kappa_g^\gamma(x) = T_x^{-1}(x_1(g^{-1}\kappa_\gamma^{-1}(x)), \dots, x_n(g^{-1}\kappa_\gamma^{-1}(x))) = (\chi(g^{-1}), m_{l_1}(g^{-1}\kappa_\gamma^{-1}(x)), \dots);$$

this implies

$$\tilde{a}_f^\gamma(x, \xi) = e^{-i(1, x_2, \dots, x_n)\cdot\xi} \int_{G_0} f(g) e^{i(\chi(g^{-1}), m_{l_1}(g^{-1}\kappa_\gamma^{-1}(x)), \dots)\cdot\xi} c_\gamma(g) d_{G_0}(g), \tag{42}$$

yielding $\tilde{a}_f^\gamma(x, \xi) \in \mathrm{C}^\infty(\tilde{U}^\gamma \times \mathbb{R}_\xi^n)$. Finally, put

$$\tilde{U}_+^\gamma = \left\{x \in \tilde{U}^\gamma : x_1 \geq 0\right\}, \qquad \tilde{U}_-^\gamma = \left\{x \in \tilde{U}^\gamma : x_1 \leq 0\right\}, \qquad \tilde{U}_*^\gamma = \left\{x \in \tilde{U}^\gamma : x_1 \neq 0\right\}.$$

We can now state the main result of this section. In what follows, $\{(\kappa_\gamma, U^\gamma)\}$ will always denote the atlas of $V_\mathbb{R} - S_\mathbb{R}^{\text{sing}}$ constructed above.

Theorem 4. *Let (G, ϱ, V) be a reductive prehomogeneous vector space whose singular set S is an irreducible hypersurface, and let Assumptions 1 and 2 be satisfied. Take $f \in \mathcal{S}(G_0)$. Then, on $V_\mathbb{R} - S_\mathbb{R}^{\text{sing}}$, the operators $\pi(f)$ are locally of the form*

$$A_f^\gamma u(x) = \int e^{ix\cdot\xi} a_f^\gamma(x,\xi)\hat{u}(\xi)d\xi, \qquad u \in \mathrm{C_c^\infty}(\tilde{U}^\gamma), \tag{43}$$

where $a_f^\gamma(x,\xi) = \tilde{a}_f^\gamma(x, T_x\xi)$, and $\tilde{a}_f^\gamma(x,\xi) \in \mathrm{S_{la}^{-\infty}}(\tilde{U}^\gamma \times \mathbb{R}^n_\xi)$ is given by (42). In particular, the kernel of the operator A_f^γ is determined by its restrictions to $\tilde{U}_*^\gamma \times \tilde{U}_*^\gamma$, and given by the oscillatory integral

$$K_{A_f^\gamma}(x,y) = \int e^{i(x-y)\cdot\xi} a_f^\gamma(x,\xi)d\xi. \tag{44}$$

Before we proceed with the proof, let us note that by restricting A_f^γ to $\tilde{U}_+^\gamma$, respectively $\tilde{U}_-^\gamma$, one obtains the continuous linear operators

$$^+A_f^\gamma : \mathrm{C_c^\infty}(\tilde{U}_+^\gamma) \longrightarrow \mathrm{C^\infty}(\tilde{U}_+^\gamma), \qquad ^-A_f^\gamma : \mathrm{C_c^\infty}(\tilde{U}_-^\gamma) \longrightarrow \mathrm{C^\infty}(\tilde{U}_-^\gamma). \tag{45}$$

By the theorem to be shown, $^+A_f^\gamma \in \mathrm{L}_b^{-\infty}(\tilde{U}_+^\gamma)$, $^-A_f^\gamma \in \mathrm{L}_b^{-\infty}(\tilde{U}_-^\gamma)$. We therefore have the following proposition.

Proposition 5. *Let (G, ϱ, V) be a reductive prehomogeneous vector space whose singular set S is an irreducible hypersurface, and let Assumptions 1 and 2 be satisfied. Then, for $f \in \mathcal{S}(G_0)$, the operators*

$$\pi(f)_{|\overline{V}_i - (\partial V_i)^{\text{sing}}} : \mathrm{C_c^\infty}(\overline{V}_i - (\partial V_i)^{\text{sing}}) \longrightarrow \mathrm{C^\infty}(\overline{V}_i - (\partial V_i)^{\text{sing}})$$

are totally characteristic pseudodifferential operators of class $\mathrm{L}_b^{-\infty}$.

We will divide the proof of Theorem 4 in several parts. To begin with, let us first state a lemma.

Lemma 3. *Let $X_1, \dots, X_{n-1}$ be elements in $\mathfrak{g}(p)$ satisfying condition (iv) with respect to a chart $(\kappa_\gamma, U^\gamma)$, and $X_n \in \mathfrak{g}$ be such that $d\chi(X_n) \neq 0$. Set $\bar{\chi}(g) = \chi(g^{-1})$, and put*

$$\Gamma(x,g) = -\begin{pmatrix} d\chi(X_1)\bar{\chi}(g) & m_{l_1}(X_1 g^{-1}\kappa_\gamma^{-1}(x)) & \dots & m_{l_{n-1}}(X_1 g^{-1}\kappa_\gamma^{-1}(x)) \\ \vdots & \vdots & \ddots & \vdots \\ d\chi(X_n)\bar{\chi}(g) & m_{l_1}(X_n g^{-1}\kappa_\gamma^{-1}(x)) & \dots & m_{l_{n-1}}(X_n g^{-1}\kappa_\gamma^{-1}(x)) \end{pmatrix},$$

as well as $\tilde{\varphi}_{\xi,x}^\gamma(g) = \varphi_{T_x^{-1}\xi}(\kappa_g^\gamma(x))$. Then

$$\begin{pmatrix} dR(X_1)\tilde{\varphi}_{\xi,x}^\gamma(g) \\ \vdots \\ dR(X_n)\tilde{\varphi}_{\xi,x}^\gamma(g) \end{pmatrix} = i\tilde{\varphi}_{\xi,x}^\gamma(g)\Gamma(x,g)\,\xi. \tag{46}$$

Moreover, $\Gamma(x,g)$ is invertible for $x \in \tilde{U}^\gamma, g \in \operatorname{supp} c_\gamma$.

Proof. First, one computes for $X \in \mathfrak{g}$, $g \in G_{\mathbb{R}}$,

$$dR(X)\tilde{\varphi}^{\gamma}_{\xi,x}(g) = \frac{d}{dh} e^{i\left(T_x^{-1}\kappa^{\gamma}_{g\,e^{hX}}(x)\right)\cdot\xi}{}_{|h=0} = i\tilde{\varphi}^{\gamma}_{\xi,x}(g)\sum_{j=1}^{n}\xi_j \frac{d}{dh}\Big(T_x^{-1}\kappa^{\gamma}_{g\,e^{hX}}(x)\Big)_{j\,|h=0}$$

$$= i\tilde{\varphi}^{\gamma}_{\xi,x}(g)\Big[\xi_1 dR(X)\bar{\chi}(g) + \sum_{j=2}^{n}\xi_j dR(X) m_{l_{j-1},\kappa_\gamma^{-1}(x)}(g)\Big].$$

By noting that

$$dR(X)\chi(e) = \frac{d}{dh}\chi(\,e^{hX}\,)_{|h=0} = d\chi(X),$$

we get $dR(X)\bar{\chi}(g) = d\chi(-X)\bar{\chi}(g)$. Similarly, one verifies the relation $dR(X)m_{l_j,\kappa_\gamma^{-1}(x)}(g) = -m_{l_j}(Xg^{-1}\kappa_\gamma^{-1}(x))$, thus obtaining (46). Now,

$$\det\Gamma(x,g) = d\chi(-Y)\bar{\chi}(g)\begin{vmatrix} m_{l_1}(-X_1g^{-1}\kappa_\gamma^{-1}(x)) & \dots & m_{l_{n-1}}(-X_1g^{-1}\kappa_\gamma^{-1}(x)) \\ \vdots & \ddots & \vdots \\ m_{l_1}(-X_{n-1}g^{-1}\kappa_\gamma^{-1}(x)) & \dots & m_{l_{n-1}}(-X_{n-1}g^{-1}\kappa_\gamma^{-1}(x)) \end{vmatrix}.$$

Hence, $\det\Gamma(x,g)$ vanishes if, and only if, the fundamental vector fields $\tilde{X}_i(m) = X_i m$, $i = 1,\dots,n-1$, do span the tangent spaces $T_m(p^{-1}(c)\cap V_{\mathbb{R}})$ at each point $m = g^{-1}\kappa_\gamma^{-1}(x)$, and if their span is not perpendicular to the hypersurface $\{m \in V_{\mathbb{R}} : m_j = 0\}$. The latter condition is equivalent to $\operatorname{grad} p(m) \notin \{m \in V_{\mathbb{R}} : m_j = 0\}$, which in turn is equivalent to the condition $\partial p/\partial m_j(m) \neq 0$ for $m = g^{-1}\kappa_\gamma^{-1}(x)$. By construction, these conditions are fulfilled for $x \in \tilde{U}^\gamma$ and $g \in \operatorname{supp} c_\gamma(g) \subset G_1^\gamma$, as a consequence of the properties (iii) and (iv) of the chart $(\kappa^\gamma, U^\gamma)$. Thus, $\Gamma(x,g)$ is invertible for $x \in \tilde{U}^\gamma$, $g \in \operatorname{supp} c_\gamma$. □

Consider now the extension of $\Gamma(x,g)$, as an endomorphism in $\mathbb{C}^1[\mathbb{R}^n_\xi]$, to the symmetric algebra $\mathrm{S}(\mathbb{C}^1[\mathbb{R}^n_\xi]) \simeq \mathbb{C}[\mathbb{R}^n_\xi]$. Since, for $x \in \tilde{U}^\gamma$, $g \in \operatorname{supp} c_\gamma$, $\Gamma(x,g)$ is invertible, its extension to $\mathrm{S}^N(\mathbb{C}^1[\mathbb{R}^n_\xi])$ is also an automorphism. Regarding the polynomials $\xi_1,\dots,\xi_n$ as a basis in $\mathbb{C}^1[\mathbb{R}^n_\xi]$, let us denote the image of the basis vector ξ_i under the endomorphism $\Gamma(x,g)$ by $\Gamma\xi_i$. Every polynomial $\xi_{i_1}\otimes\cdots\otimes\xi_{i_N} \equiv \xi_{i_1}\dots\xi_{i_n}$ can then be written as a linear combination

$$\xi^\alpha = \sum_\beta \Lambda^\alpha_\beta(x,g)\Gamma\xi_{\beta_1}\cdots\Gamma\xi_{\beta_{|\alpha|}}, \tag{47}$$

where the $\Lambda^\alpha_\beta(x,g)$ are C^∞ functions on $\tilde{U}^\gamma \times \operatorname{supp} c_\gamma$ which, in addition, are rational in g, and $\Gamma\xi_k = -i\tilde{\varphi}^\gamma_{-\xi,x}(g)dR(X_k)\tilde{\varphi}^\gamma_{\xi,x}(g)$, by (46).

Lemma 4. *For arbitrary indices $\beta_1,\dots,\beta_r$, one has*

$$\begin{aligned} i^r\tilde{\varphi}^\gamma_{\xi,x}(g)\Gamma\xi_{\beta_1}\cdots\Gamma\xi_{\beta_r} &= dR(X_{\beta_1}\cdots X_{\beta_r})\tilde{\varphi}^\gamma_{\xi,x}(g) \\ &\quad + \sum_{s=1}^{r-1}\sum_{\alpha_1,\dots,\alpha_s} d^{\beta_1,\dots,\beta_r}_{\alpha_1,\dots,\alpha_s}(x,g)dR(X_{\alpha_1}\cdots X_{\alpha_s})\tilde{\varphi}^\gamma_{\xi,x}(g), \end{aligned} \tag{48}$$

where the coefficients $d^{\beta_1,\dots,\beta_r}_{\alpha_1,\dots,\alpha_s}(x,g) \in \mathrm{C}^\infty(\tilde{U}^\gamma \times \operatorname{supp} c_\gamma)$ are at most of exponential growth in g, and independent of ξ.

Proof. For $r = 1$ one has $i\tilde{\varphi}^\gamma_{\xi,x}(g)\Gamma\xi_k = dR(X_k)\tilde{\varphi}^\gamma_{\xi,x}(g)$, and differentiating the latter equation with respect to X_j we obtain, by taking (47) into account,

$$-\tilde{\varphi}^\gamma_{\xi,x}(g)\Gamma\xi_j\Gamma\xi_k = dR(X_jX_k)\tilde{\varphi}^\gamma_{\xi,x}(g) - \sum_{l,r=1}^{n}(dR(X_j)\Gamma_{kl})(x,g)\Lambda^l_r(x,g)dR(X_r)\tilde{\varphi}^\gamma_{\xi,x}(g).$$

Hence, the assertion of the lemma is correct for $r = 1, 2$. Now, assume that it holds for $r \leq N$. Setting $r = N$ in equation (48), and differentiating with respect to X_k, yields for the left hand side

$$i^{N+1}\tilde{\varphi}^{\gamma}_{\xi,x}(g)\Gamma\xi_k\Gamma\xi_{\beta_1}\cdots\Gamma\xi_{\beta_N} + i^N\tilde{\varphi}^{\gamma}_{\xi,x}(g)\Big(\sum_{l,q=1}^{n}(dR(X_k)\Gamma_{\beta_1 l})(x,g)\Lambda^l_q(x,g)\Gamma\xi_q\Big)\Gamma\xi_{\beta_2}\cdots\Gamma\xi_{\beta_N}$$

plus additional terms. By assumption, we can apply (48) to the products $\Gamma\xi_q\Gamma\xi_{\beta_2}\cdots\Gamma\xi_{\beta_N}, \dots$ of at most N factors. This proves the lemma. $\square$

We are now in position to prove Theorem 4.

Proof of Theorem 4. We first show that $\tilde{a}^{\gamma}_f(x,\xi) \in \mathrm{S}^{-\infty}(\tilde{U}^{\gamma} \times \mathbb{R}^n_{\xi})$. As already noted, $\tilde{a}^{\gamma}_f(x,\xi) \in \mathrm{C}^{\infty}(\tilde{U}^{\gamma} \times \mathbb{R}^n_{\xi})$. While differentiation with respect to ξ does not alter the growth properties of $\tilde{a}^{\gamma}_f(x,\xi)$, differentiation with respect to x yields additional powers in ξ. Now, as an immediate consequence of equations (47) and (48), one computes

$$\tilde{\varphi}^{\gamma}_{\xi,x}(g)(1+\xi^2)^N = \sum_{r=0}^{2N}\sum_{|\alpha|=r} b^N_{\alpha}(x,g)dR(X^{\alpha})\tilde{\varphi}^{\gamma}_{\xi,x}(g), \tag{49}$$

where the coefficients $b^N_{\alpha}(x,g) \in \mathrm{C}^{\infty}(\tilde{U}^{\gamma} \times \operatorname{supp} c_{\gamma})$ are rational expressions in the matrix coefficients of g. Now, $(\partial^{\alpha}_{\xi}\,\partial^{\beta}_x\,\tilde{a}^{\gamma}_f)(x,\xi)$ is a finite sum of terms of the form

$$\xi^{\delta}e^{-i(1,x_1,\dots,x_n)\cdot\xi}\int_{G_0} f(g)d_{\delta\beta}(x,g)\tilde{\varphi}^{\gamma}_{\xi,x}(g)c_{\gamma}(g)d_{G_0}(g),$$

the functions $d_{\delta\beta}(x,g) \in \mathrm{C}^{\infty}(\tilde{U}^{\gamma}\times\operatorname{supp} c_{\gamma})$ being at most of exponential growth in g. Making use of equation (49), and integrating by parts according to Proposition 1, we finally obtain for arbitrary α, β the estimate

$$|(\partial^{\alpha}_{\xi}\,\partial^{\beta}_x\,\tilde{a}^{\gamma}_f)(x,\xi)| \leq \frac{1}{(1+\xi^2)^N}C_{\alpha,\beta,\omega} \qquad x \in \omega,$$

where ω denotes an arbitrary compact set in $\tilde{U}^{\gamma}$ and $N = 1, 2, \dots$. This proves that $\tilde{a}^{\gamma}_f(x,\xi) \in \mathrm{S}^{-\infty}(\tilde{U}^{\gamma} \times \mathbb{R}^n_{\xi})$. Equation (43) is an immediate consequence of Fourier Inversion Formula, and it remains to show that $\tilde{a}^{\gamma}_f(x,\xi)$ satisfies the lacunary condition (18). We have $a^{\gamma}_f(x,\xi) = \tilde{a}^{\gamma}_f(x, x_1\xi_1, \xi')$, and by the estimates

$$(1+(x_1\xi_1)^2+\cdots+\xi_n^2)^{-N} \leq \begin{cases} \langle\xi\rangle^{-N} & \text{for } |x_1| \geq 1 \\ |x_1|^{-N}\langle\xi\rangle^{-N} & \text{for } 0 < |x_1| < 1 \end{cases}$$

one concludes that $a^{\gamma}_f(x,\xi) \in \mathrm{S}^{-\infty}(\tilde{U}^{\gamma}_* \times \mathbb{R}^n_{\xi})$, which is also clear from Theorem 2. Hence, $A^{\gamma}_{f\,|\tilde{U}^{\gamma}_*} \in \mathrm{L}^{-\infty}(\tilde{U}^{\gamma}_*)$, by Equation (43). The Schwartz kernel of $A^{\gamma}_{f\,|\tilde{U}^{\gamma}_*}$ is given by the oscillatory integral

$$\int e^{i(x-y)\cdot\xi}a^{\gamma}_f(x,\xi)\,đ\xi, \qquad x_1 \neq 0.$$

On the other hand, since on $\tilde{U}^{\gamma}_+$, respectively $\tilde{U}^{\gamma}_-$, $A^{\gamma}_f u$ only depends on the restriction of u to $\tilde{U}^{\gamma}_+$, respectively $\tilde{U}^{\gamma}_-$, we must have

$$\operatorname{supp}\mathcal{K}_{A^{\gamma}_f} \subset (\tilde{U}^{\gamma}_+ \times \tilde{U}^{\gamma}_+) \cup (\tilde{U}^{\gamma}_- \times \tilde{U}^{\gamma}_-),$$

where $\mathcal{K}_{A_f^\gamma} \in \mathcal{D}'(\tilde{U}^\gamma \times \tilde{U}^\gamma)$ is the Schwartz kernel of A_f^γ as a continuous linear operator from $\mathrm{C^\infty_c}(\tilde{U}^\gamma)$ to $\mathrm{C}^\infty(\tilde{U}^\gamma)$. Consequently,

$$\int e^{i(x_1-y_1)\xi_1} \tilde{a}_f^\gamma(x, x_1\xi_1, \xi')\, đ\xi_1,$$

which is a C^∞ function for $x_1 > 0, y_1 < 0$, respectively $x_1 < 0, y > 0$, must vanish then. By making the substitution $t = y_1/x_1 - 1$, we arrive at the condition

$$M\tilde{a}_f^\gamma(x, \xi'; t) = \int e^{-it\xi_1} \tilde{a}_f^\gamma(x, \xi)\, đ\xi_1 = 0 \qquad \text{for } t < -1,\, x \in \tilde{U}_*^\gamma; \tag{50}$$

since $\tilde{a}_f^\gamma(x,\xi) \in \mathrm{C}^\infty(\tilde{U}^\gamma \times \mathbb{R}^n_\xi)$ is rapidly falling in ξ, (50) must hold also for $x \in \tilde{U}^\gamma$, meaning that $\tilde{a}_f^\gamma(x,\xi)$ satisfies the lacunary condition (18). Finally, consider the operators introduced in (45). They extend to continuous linear operators from $\dot{\mathcal{E}}'(\tilde{U}_\pm^\gamma)$ to $\dot{\mathcal{D}}'(\tilde{U}_\pm^\gamma)$, and according to [7], Lemma 4.1, their Schwartz kernels are given by their restrictions to $\mathrm{Int}(\tilde{\mathrm{U}}_+^\gamma) \times \mathrm{Int}(\tilde{\mathrm{U}}_+^\gamma)$, respectively $\mathrm{Int}(\tilde{\mathrm{U}}_-^\gamma) \times \mathrm{Int}(\tilde{\mathrm{U}}_-^\gamma)$. Consequently, the kernel $\mathcal{K}_{A_f^\gamma}$ must be determined by its restriction to $\tilde{U}_*^\gamma \times \tilde{U}_*^\gamma$, and hence, by the oscillatory integral (44). This completes the proof of Theorem 4. ☐

As a consequence of Theorem 4 we can write the kernel of A_f^γ in the form

$$\begin{aligned} K_{A_f^\gamma}(x,y) &= \int e^{i(x-y)\cdot\xi} a_f^\gamma(x,\xi) đ\xi = \int e^{i(x-y)\cdot T_x^{-1}\xi} \tilde{a}_f^\gamma(x,\xi) |\det (T_x^{-1})'(\xi)| đ\xi \\ &= \frac{1}{|x_1|} \tilde{A}_f^\gamma(x, T_x^{-1}(x-y)), \qquad x_1 \neq 0, \end{aligned} \tag{51}$$

where $\tilde{A}_f^\gamma(x,y)$ denotes the inverse Fourier transform of $\tilde{a}_f^\gamma(x,\xi)$,

$$\tilde{A}_f^\gamma(x,y) = \int e^{iy\cdot\xi} \tilde{a}_f^\gamma(x,\xi)\, đ\xi.$$

Since, for $x \in \tilde{U}^\gamma$, $\tilde{a}_f^\gamma(x,\xi)$ is rapidly falling in ξ, it follows that $\tilde{A}_f^\gamma(x,y) \in \mathcal{S}(\mathbb{R}^n_y)$, the Fourier transform being an isomorphism on the Schwartz space. Consider the restriction of the atlas $\{(\kappa_\gamma, U^\gamma)\}$ to $V_\mathbb{R} - S_\mathbb{R}$; by (51), the restriction of the kernel of the operator $\pi(f)$ to the diagonal is given by the family of functions

$$k_f^\gamma(m) = K_{A_f^\gamma}(\kappa_\gamma(m), \kappa_\gamma(m)) = \frac{1}{|p(m)|} \tilde{A}_f^\gamma(\kappa_\gamma(m), 0),$$

and we denote the corresponding density on $V_\mathbb{R} - S_\mathbb{R}$ by k_f.

Proposition 6. *Let (G_C, ϱ, V) be a reductive prehomogeneous vector space whose singular set is an irreducible hypersurface, and for which Assumptions* 1 *and* 2 *are satisfied. Let $V_\mathbb{R} - S_\mathbb{R} = V_1 \cup \cdots \cup V_l$ be the decomposition of $V_\mathbb{R} - S_\mathbb{R}$ into its connected components, and define the family of functions*

$$k_f^\gamma(m,s) = |p(m)|^s \tilde{A}_f^\gamma(\kappa_\gamma(m), 0).$$

Denote the corresponding density on $V_\mathbb{R} - S_\mathbb{R}$ by $k_{f,s}$. Then, for $\varphi \in \mathrm{C^\infty_c}(V_\mathbb{R} - S_\mathbb{R}^{\mathrm{sing}})$, the integrals

$$\int_{V_\mathbb{R} - S_\mathbb{R}} \varphi\, k_{f,s} = \sum_{i=1}^{l} \int_{V_i} \varphi\, k_{f,s} \tag{52}$$

converge for $\mathrm{Re}\, s > 0$*, and can be continued analytically to meromorphic functions in $s \in \mathbb{C}$, satisfying functional equations of the form* (12).

Proof. Let χ_γ be a partition of unity subordinated to the covering $\{U^\gamma\}$. Identifying the functions $k_f^\gamma(m,s)$ with the densities $k_f^\gamma(m,s)\,dm$, we obtain

$$\sum_{i=1}^{l}\int_{V_i}\varphi\, k_{f,s} = \sum_{i=1}^{l}\sum_{\gamma}\int_{V_i\cap U^\gamma}\varphi(m)\chi_\gamma(m)k_f^\gamma(m,s)\,dm$$
$$= \sum_{i=1}^{l}\sum_{\gamma}\int_{V_i\cap U^\gamma}\varphi(m)\chi_\gamma(m)\tilde{A}_f^\gamma(\kappa_\gamma(m),0)|p(m)|^s dm.$$

Since $\varphi(m)\chi_\gamma(m)\tilde{A}_f^\gamma(\kappa_\gamma(m),0)\in \mathrm{C_c^\infty}(V_\mathbb{R}-S_\mathbb{R}^{\mathrm{sing}})$, the assertion follows with Theorem 1. Note that there exists a finite covering of $\operatorname{supp}\varphi$ by the U^γ, so that, in particular, the sum over γ is finite. □

Let us illustrate this in the situation of Example 3. By (39), one has, in this case, the functional equations

$$\begin{pmatrix} \int_{V_+^*}|p^*(\xi)|^{s-\frac{n}{2}}\hat{\varphi}*\widehat{\chi_\gamma\tilde{A}_f^\gamma}(\xi)\,d\xi \\ \int_{V_-^*}|p^*(\xi)|^{s-\frac{n}{2}}\hat{\varphi}*\widehat{\chi_\gamma\tilde{A}_f^\gamma}(\xi)\,d\xi \end{pmatrix} = (2\pi)^n C(s)\begin{pmatrix} \int_{V_+}k_{f,-s}^\gamma\chi_\gamma(m)\varphi(m)\,dm \\ \int_{V_-}k_{f,-s}^\gamma\chi_\gamma(m)\varphi(m)\,dm \end{pmatrix},$$

where we set $\tilde{A}_f^\gamma(m)=\tilde{A}_f^\gamma(\kappa_\gamma(m),0)$, and $\varphi\in \mathrm{C_c^\infty}(V_\mathbb{R}-S_\mathbb{R}^{\mathrm{sing}})$.

Remark 4. We close this section by noting that a trace of $\pi(f)$ can be defined in the situation of Propositions 5 and 6 by a regularization procedure analogous to the one in Proposition 4. Alternatively, one could also use a method due to Hadamard, which would essentially yield the b-trace introduced by Melrose in his work on totally characteristic pseudodifferential operators.

7. Strongly elliptic differential operators and kernels of holomorphic semigroups

As an application of the results of the previous sections, we study the holomorphic semigroup generated by a strongly elliptic differential operator associated with the left regular representation $(\pi,\mathrm{C}_0(V_\mathbb{R}))$ of a prehomogeneous vector space (G,ϱ,V). Let π be a continuous representation of a Lie group G on a Banach space $\mathcal{B}$, $\mathfrak{g}$ the Lie-algebra of G, and $X_1,\dots,X_d$ a basis of $\mathfrak{g}$. A differential operator

$$\Omega=\sum_{|\alpha|\leq l}(-i)^{|\alpha|}c_\alpha d\pi(X^\alpha)$$

associated with the representation π is called *strongly elliptic* if, for all $\xi\in\mathbb{R}^d$, the relation $\mathrm{Re}\sum_{|\alpha|=l}c_\alpha\xi^\alpha\geq\kappa|\xi|^l$ is satisfied, where $\kappa>0$ is some positive number. By Langlands [6], the closure of a strongly elliptic operator generates a strongly continuous holomorphic semigroup of bounded operators on $\mathcal{B}$ given by

$$S_t=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda\mathbf{1}+\overline{\Omega})^{-1}d\lambda,$$

where Γ is a appropriate path in $\mathbb{C}$ coming from infinity and going to infinity, such that $\lambda\notin\sigma(\overline{\Omega})$ for $\lambda\in\Gamma$. Here $|\arg t|<\alpha$ for an appropriate $\alpha\in(0,\pi/2]$, and the integral converges uniformly with respect to the operator norm. The proof of this fact essentially relies on the verification of a criterion of Hille [2] in the theory of strongly continuous semigroups. The subgroup S_t can be characterized by a convolution semigroup of complex measures μ_t on G according to

$$S_t = \int_G \pi(g) d\mu_t(g),$$

π being measurable with respect to the measures μ_t. The measures μ_t are absolutely continuous with respect to Haar measure d_G on G, and denoting by $K_t(g) \in L^1(G, d_G)$ the corresponding Radon-Nikodym derivative, one has

$$S_t = \pi(K_t) = \int_G K_t(g)\pi(g) d_G(g).$$

The function $K_t(g) \in \mathrm{L}^1(G, d_G)$ is analytic in t and g, and universal for all Banach representations. Moreover, it is supported on the identity component G^0 of G. In the following, it will be called the *Langlands kernel* of the holomorphic semigroup S_t. Though, in general, the Langlands kernel is not explicitely known, it satisfies the following L^1- and L^∞-bounds. A detailed exposition of these facts can be found in [11], Chapter III (see, in particular, pages 152, 166, and 209).

Theorem 5 (Robinson)**.** *Let $(L, \mathrm{C}^\infty(G))$ be the left regular representation of G. Then, for each $\kappa \geq 0$, there exist constants $a, b, c > 0$, and $\omega \geq 0$ such that*

$$\int_{G^0} |dL(X^\alpha)\, \partial_t^\beta\, K_t(g)| e^{\kappa|g|}\, d_{G^0}(g) \leq a b^{|\alpha|} c^\beta |\alpha|!\, \beta! (1 + t^{-\beta - |\alpha|/l}) e^{\omega t},$$

for all $t > 0$, $\beta = 0, 1, 2, \dots$ and multiindices α. Furthermore,

$$|dL(X^\alpha)\, \partial_t^\beta\, K_t(g)| \leq a b^{|\alpha|} c^\beta |\alpha|!\, \beta! (1 + t^{-\beta - (|\alpha|+d+1)/l}) e^{\omega t} e^{-\kappa|g|},$$

for all $g \in G^0$, where $d = \dim \mathfrak{g}$, and l denotes the order of Ω.

Let (G, ϱ, V) be a reductive prehomogeneous vector space defined over $\mathbb{R}$, and denote by $(\pi, \mathrm{C}_0(V_\mathbb{R}))$ the regular representation of $G_\mathbb{R}$ on $V_\mathbb{R}$. Assume that Ω is a strongly elliptic operator associated with π, and consider the corresponding holomorphic semigroup $S_t = \pi(K_t)$ with Langlands kernel K_t. It induces a continuous linear mapping

$$S_t : \mathrm{C_c^\infty}(\mathbb{R}^n) \longrightarrow \mathrm{C}_0(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n), \tag{53}$$

and since $K_t \in \mathcal{S}(G_0)$ by the previous theorem, the results of the previous sections concerning the Schwartz kernel of the operator (53) are applicable. We will illustrate this for the classical heat kernel and the simplest prehomogeneous vector space of Example 1.

Thus, let $G = \mathbb{C}^*$, $V = \mathbb{C}$, $G_\mathbb{R} = \mathrm{GL}(1, \mathbb{R}) = \mathbb{R}^+_*$, and $V_\mathbb{R} = \mathbb{R}$. Let $a = 1$ be a basis of $\mathfrak{g} = \mathbb{R}$. The action of $d\pi(a)$ on the Gårding subspace $\mathrm{C}_0(\mathbb{R})_\infty$ is given by the vector field

$$d\pi(a)\varphi(x) = \operatorname{grad}\varphi(x) \cdot \Big(\frac{d}{dh}\, \mathrm{e}^{-h}\, x\Big)_{\big|_{h=0}} = -x \frac{d}{dx}\varphi(x).$$

The Casimir operator $\Omega = \big(d\pi(a)\big)^2 \in \mathfrak{U}$ of the considered representation therefore reads

$$\Omega = x^2 \frac{d^2}{dx^2} + x \frac{d}{dx},$$

and constitutes a differential operator of Euler type. Let $d_{G_0}(x) = dx/x$ be Haar measure on $\mathrm{GL}(1, \mathbb{R})$. Denoting by S_t the holomorphic semigroup generated by $\overline{\Omega}$, and by $K_t(x)$ the corresponding Langlands kernel, we get

$$S_t(\varphi) = \int_{\mathbb{R}^+_*} K_t(x)\pi(x)\varphi \frac{dx}{x}, \qquad \varphi \in \mathrm{C}_0(\mathbb{R}).$$

In order to compute $K_t(x)$ explicitely, we consider the Banach representation $(\pi, \mathrm{C}_0(\mathbb{R}^+_*))$, introducing on $\mathbb{R}^+_*$ the new coordinate $x = e^r$, $r \in \mathbb{R}$. The Casimir operator is then given by d^2/dr^2, and the Langlands kernel coincides with the classical heat kernel $K_t(r) = (2\pi t)^{-1/2} \exp(-r^2/4t)$. By transforming back, we get for $K_t(x)$ the universal expression

$$K_t(x) = \frac{1}{\sqrt{2\pi t}} \, \mathrm{e}^{-(\log x)^2/4t} \, .$$

Since the Langlands kernel is explicitely known in this case, it is illustrative to give a direct proof of Theorem 3. Since G_0 is unimodular, one has

$$s_t(x,\xi) = e^{-ix\xi} \frac{1}{2\pi t} \int\limits_0^\infty e^{-(\log y)^2/4t} e^{ixy\xi} \frac{dy}{y} = e^{-ix\xi} \hat{f}_t(-x\xi),$$

where we defined the function $f_t(y)$ by $K_t(y)y^{-1}$ for $y > 0$, and set it equal 0 for $y \leq 0$. Clearly, $f_t \in \mathcal{S}(\mathbb{R})$. Indeed, $K_t(y)y^m \in \mathrm{C}^\infty(\mathbb{R}^+_*)$ for all $m \in \mathbb{Z}$, and by the substitution $y = e^r$, we obtain

$$K_t(y)y^m = \frac{1}{\sqrt{2\pi t}} e^{-r^2/4t + mr} = \frac{1}{\sqrt{2\pi t}} e^{-(r/(2\sqrt{t}) - \sqrt{t}m)^2 + m^2 t};$$

for $y \to 0$ and $y \to \infty$, $K_t(y)y^m$ goes to zero together with all its derivatives, implying $|y^\alpha \, \partial_y^\beta \, f_t(y)| < \infty$ for arbitrary multiindices α, β. Hence we deduce $s_t(x,\xi) \in \mathrm{S}^1(\mathbb{R}_y \times \mathbb{R}_\xi)$ for arbitrary $x \in \mathbb{R}$, and $s_t(x,\xi) \in \mathrm{S}^{-\infty}(\mathbb{R}_y \times \mathbb{R}_\xi)$ for $x \neq 0$. Since

$$\partial_\xi^\alpha \, \partial_x^\beta \, \hat{f}_t(-x\xi) = \sum_{\alpha_1 + \alpha_2 = \alpha} \frac{\alpha!}{\alpha_1 \alpha_2} \, \partial_\chi^{\beta+\alpha_1} \, \hat{f}_t(\chi)_{|\chi = -x\xi} (-x)^{\alpha_1} \beta \cdots (\beta - \alpha_2 + 1)(-\xi)^{\beta - \alpha_2},$$

we finally obtain $s_t(x,\xi) \in \mathrm{S}^{-\infty}(\mathbb{R}_x - \{0\} \times \mathbb{R}_y \times \mathbb{R}_\xi)$. Thus, $S_t : \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ is given by the family of oscillatory integrals

$$(S_t \varphi)(x) = \int e^{i(x-y)\xi} s_t(x,\xi) \varphi(y) dy đ\xi.$$

In particular, $S_t \in \mathrm{L}^{-\infty}(\mathbb{R} - \{0\})$, proving Theorem 3 in the present situation. Now, an easy computation shows that if we define

$$S_t(x,y) = \begin{cases} K_t(xy^{-1})|y|^{-1} & \text{for } xy^{-1} > 0 \\ 0 & \text{for } xy^{-1} \leq 0, \end{cases}$$

we get for S_t the representation

$$(S_t\varphi)(x) = \begin{cases} \int\limits_{-\infty}^{\infty} S_t(x,y)\varphi(y)dy, & x \neq 0, \\ \|K_t\|_{L^1} \, \varphi(0), & x = 0. \end{cases}$$

Lemma 2 then implies that the Schwartz kernel of $S_t : \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ is given by the function $S_t(x,y)$. Indeed, for $x \neq 0$,

$$\int e^{i(x-y)\xi} s_t(x,\xi) đ\xi = \int \int e^{i(z-y)\xi} f_t(zx^{-1}) \frac{1}{|x|} \, dz đ\xi = f_t(yx^{-1})|x|^{-1} = S_t(x,y).$$

Following the discussion in Section 5, we define the auxiliary symbol

$$\tilde{a}_t(x,\xi) = e^{-i\xi} \hat{f}_t(-\xi) \in \mathrm{S}^{-\infty}(\mathbb{R}_x \times \mathbb{R}_\xi),$$

so that $s_t(x,\xi) = \tilde{a}_t(x, x\xi)$. The inverse Fourier transform

$$\tilde{A}_t(x,y) = \int e^{iy\xi} \tilde{a}_t(x,\xi) đ\xi = f_t(1-y)$$

is a Schwartz function which vanishes for $y \geq 1$ together with all its derivatives, implying that $\tilde{a}_t(x,\xi)$ satisfies the lacunary condition (18). For $x \neq 0$ one computes

$$S_t(x,y) = \int e^{i(x-y)\xi} \tilde{a}_t(x, x\xi) đ\xi = \frac{1}{|x|} \tilde{A}_t\big(x, \frac{x-y}{x}\big), \qquad x \neq 0.$$

In concordance with Proposition 4, the restriction $s_t(x) = S_t(x,x)$ of S_t to the diagonal defines a homogeneous distribution of degree -1 on $\mathbb{R} - \{0\}$, which has an extension to $\mathbb{R}$ given by (38). Note that $\tilde{A}_t(x,0) = K_t(1) = (2\pi t)^{-1/2}$. Finally, setting $s_{t,\zeta}(x) = (2\pi t)^{-1/2}|x|^\zeta$, we obtain with (36) the functional equation

$$\int_{-\infty}^{\infty} s_{t,\zeta-1} \hat{\varphi}(\xi)\, d\xi = 2(2\pi)^{-\zeta}\Gamma(\zeta) \cos\frac{\pi\zeta}{2} \int_{-\infty}^{\infty} s_{t,-\zeta}(x)\varphi(x)\, dx, \qquad \varphi \in \mathcal{S}(\mathbb{R}).$$

References

[1] J. Elstrodt, *Maß- und Integrationstheorie*, Springer-Verlag, Berlin, 1999.

[2] E. Hille and R. S. Phillips, *Functional analysis and semi–groups*, American Mathamatical Society Colloqium Publications, 1957.

[3] L. Hörmander, *The analysis of linear partial differential operators*, vol. I, Springer–Verlag, Berlin, Heidelberg, New York, 1983.

[4] ———, *The analysis of linear partial differential operators*, vol. III, Springer–Verlag, Berlin, Heidelberg, New York, 1985.

[5] T. Kimura, *Introduction to prehomogeneous vector spaces*, American Mathematical Society, Providence, Rhode Island, 2003.

[6] R. P. Langlands, *Semi–groups and representations of Lie–groups*, Doctoral thesis, Yale University, unpublished, 1960.

[7] R. Melrose, *Transformation of boundary problems*, Acta Math. **147** (1982), 149–236.

[8] J. W. Milnor, *Morse theory*, Princeton University Press, Princeton, New Jersey, 1973.

[9] A. N. Parshin and I. R. Shafarevich (eds.), *Algebraic geometry IV*, Springer-Verlag, Berlin, Heidelberg, New York, 1994.

[10] P. Ramacher, *Analysis on real affine G-varieties*, math.RT/0309089, to appear in J. of Lie Theory, 2004.

[11] D. W. Robinson, *Elliptic operators and Lie groups*, Oxford University Press, Oxford, 1991.

[12] M. A. Shubin, *Pseudodifferential operators and spectral theory*, 2nd edition, Springer–Verlag, Berlin, Heidelberg, New York, 2001.

[13] N. R. Wallach, *Real reductive groups*, vol. I, Academic Press, Inc., Boston, San Diego, New York, 1988.

[14] G. Warner, *Harmonic analysis on semi-simple Lie groups*, vol. I, Springer–Verlag, Berlin, Heidelberg, New York, 1972.

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