# Stability and Hermitian-Einstein metrics for vector bundles on framed manifolds

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#### Framed stability and Hermitian-Einstein metrics

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### Let (X, g) be a compact Kähler manifold of complex dimension n and $\mathcal{E}$ a torsion-free coherent sheaf on X.

▶ The q-degree of  $\mathcal{E}$  is defined as

$$\deg_g(\mathcal{E}) = (c_1(\mathcal{E}) \cup [\omega_g]^{n-1}) \cap [X]$$
$$= \int_X c_1(\mathcal{E}) \wedge \omega_g^{n-1}.$$

▶ If  $rank(\mathcal{E}) > 0$ , the *g*-slope of  $\mathcal{E}$  is defined as

$$\mu_g(\mathcal{E}) = \frac{\deg_g(\mathcal{E})}{\operatorname{rank}(\mathcal{E})}$$

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# Stability

### Definition

 $\triangleright$   $\mathcal{E}$  is called *g*-semistable if

$$\mu_g(\mathcal{F}) \leqslant \mu_g(\mathcal{E})$$

holds for every coherent subsheaf  $\mathcal{F}$  of  $\mathcal{E}$  with  $0 < \operatorname{rank}(\mathcal{F})$ .

If, moreover,

$$\mu_g(\mathcal{F}) < \mu_g(\mathcal{E})$$

holds for every coherent subsheaf  $\mathcal{F}$  of  $\mathcal{E}$  with  $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$ , then  $\mathcal{E}$  is called g-stable.

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### Definition

 ${\mathcal E}$  is called g-polystable if  ${\mathcal E}$  is g-semistable and  ${\mathcal E}$  decomposes as a direct sum

$$\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m$$

of g-stable coherent subsheaves  $\mathcal{E}_1,\ldots,\mathcal{E}_m$  of the same g-slope  $\mu_g(\mathcal{E}_i)=\mu_g(\mathcal{E}).$ 

### Remark

These notions are also defined for a holomorphic vector bundle E:

Consider  $\mathcal{E} = \mathcal{O}_X(E)$ .

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These notions are also defined for a holomorphic vector bundle E:

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### Lemma

Every g-stable holomorphic vector bundle E is simple, i. e.

$$\operatorname{End}(E) = \mathbb{C} \cdot \operatorname{id}_E$$
.

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## Hermitian-Einstein metrics

Let E be a holomorphic vector bundle on X.

### Definition

A Hermitian metric h in E is called a q-Hermitian-Einstein metric if

$$i\Lambda_g F_h = \lambda \operatorname{id}_E \quad \text{with } \lambda \in \mathbb{R},$$

### where

- $i\Lambda_q = \text{contraction with } \omega_q$ ,
- $ightharpoonup F_h = \text{Chern curvature form of } (E, h).$

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Remark

From the Einstein equation

$$i\Lambda_q F_h = \lambda \operatorname{id}_E,$$

it follows by taking the trace and integrating, that we must have

$$\lambda = \frac{2\pi\mu_g(E)}{(n-1)!\operatorname{vol}_g(X)}.$$

# Kobayashi-Hitchin correspondence

Let E be a holomorphic vector bundle.

Theorem (S. Kobayashi '82, Lübke '83)

If E admits a g-Hermitian-Einstein metric, then E is g-polystable.

In particular, if E is irreducible, then it is g-stable.

Theorem (Donaldson '83–'87, Uhlenbeck/Yau '86) If E is g-stable, then E admits a unique (up to a constant multiple) g-Hermitian-Einstein metric.

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Proof.

- ▶ Start with a fixed "background metric"  $h_0$ .
- ▶ Construct a family  $(h_t)$  of Hermitian metrics solving the evolution equation

$$h_t^{-1}\dot{h}_t = -(i\Lambda_g F_{h_t} - \lambda \operatorname{id}_E)$$

for all finite values of the time parameter t, where  $\lambda$  is as before.

▶ In case of convergence as  $t \to \infty$ : The limit is a Hermitian-Einstein metric.

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- ▶ In case of divergence:
  - Construct a "destabilizing subsheaf" contradicting the stability hypothesis, i. e. a coherent subsheaf  $\mathcal{F}$  of  $\mathcal{E} = \mathcal{O}_X(E)$  with  $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$  and

$$\mu_g(\mathcal{F}) \geqslant \mu_g(\mathcal{E}).$$

This is done by first constructing a weakly holomorphic subbundle of E, i. e. a section  $\pi \in L^2_1(\operatorname{End}(E))$  satisfying

$$\pi = \pi^* = \pi^2$$
 and  $(\mathrm{id}_E - \pi) \circ \bar{\partial}\pi = 0$ 

and applying a regularity theorem of Uhlenbeck-Yau.

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### Definition

- ► A framed manifold or logarithmic pair is a pair (X, D) consisting of
  - ightharpoonup a compact complex manifold X and
  - ightharpoonup a smooth divisor D in X.
- ▶ A framed manifold (X, D) is called canonically polarized if  $K_X \otimes [D]$  is ample.

## Example

 $(\mathbb{P}^n,V)$  is canonically polarized if  $V\subset\mathbb{P}^n$  is a smooth hypersurface of degree  $\geqslant n+2$ .

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Classical result:

Theorem (Yau '78)

If  $K_X$  is ample, there exists a unique (up to a constant multiple) Kähler-Einstein metric on X with negative Ricci curvature.

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If (X,D) is canonically polarized, there exists a unique (up to a constant multiple) complete Kähler-Einstein metric on  $X':=X\setminus D$  with negative Ricci curvature.

### Remark

- lacktriangle We call this metric the Poincaré-type metric on X'.
- ▶ Choose local coordinates  $(\sigma, z^2, ..., z^n)$  such that D is given by  $\sigma = 0$ . Then in these coordinates, we have

$$\omega_{\mathsf{Poin}} \sim 2i \left( \frac{d\sigma \wedge d\bar{\sigma}}{|\sigma|^2 \log^2(1/|\sigma|^2)} + \sum_{k=2}^n dz^k \wedge d\bar{z}^k \right).$$

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(Cheng-Yau '80, R. Kobayashi '84, Tian-Yau '87, ...)

### **Definition**

A local quasi-coordinate map is a holomorphic map

$$V \longrightarrow X', \quad V \subset \mathbb{C}^n$$
 open

which is of maximal rank everywhere. In this case, V together with the Euclidean coordinates of  $\mathbb{C}^n$  is called a local quasi-coordinate system.

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### Theorem

X' together with the Poincaré-type metric is of bounded geometry, i. e. there is an (infinite) family  $\mathcal{V} = \{(V; v^1, \dots, v^n)\}$  of local quasi-coordinate systems such that:

- ightharpoonup X' is covered by the images of the V in  $\mathcal V$ .
- ▶ There is an open neighbourhood U of D such that  $X \setminus U$  is covered by the images of finitely many V which are coordinate systems in the ordinary sense.
- ▶ Every V contains an open ball of radius  $\frac{1}{2}$ .
- ► The coefficients of the Poincaré-type metric and their derivatives in quasi-coordinates are uniformly bounded.

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$$(\Delta^n; z^1, \dots, z^n)$$
 on  $U \subset X$ 

such that

$$D \cap U = \{ z^1 = 0 \}.$$

Then the quasi-coordinates are  $(R_{\rm p}(0) \times \Lambda^{n-1}, n^1, n^1)$  with  $\frac{1}{2} < \frac{1}{2}$ 

$$(B_R(0) \times \Delta^{n-1}; v^1, \dots, v^n)$$
 with  $\frac{1}{2} < R < 1$  such that

$$v^{1} = \frac{w^{1} - a}{1 - aw^{1}}, \text{ where } z^{1} = \exp\left(\frac{w^{1} + 1}{w^{1} - 1}\right)$$

and

$$v^i = w^i = z^i$$
 for  $2 \leqslant i \leqslant n$ ,

where a varies over real numbers in  $\Delta$  close to 1.

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Theorem (Schumacher '98)

There exists  $0 < \alpha \leqslant 1$  such that for all  $k \in \{0, 1, \ldots\}$  and  $\beta \in (0, 1)$ , the volume form of the Poincaré-type metric is of the form

$$\frac{2\Omega}{||\sigma||^2\log^2(1/||\sigma||^2)}\left(1+\frac{\nu}{\log^\alpha(1/||\sigma||^2)}\right),$$

#### where

- $ightharpoonup \Omega$  is a smooth volume form on X,
- $ightharpoonup \sigma$  is a canonical section of [D],  $||\cdot||$  is a norm in [D],
- $\blacktriangleright \nu$  lies in the Hölder space of  $\mathcal{C}^{k,\beta}$  functions in quasi-coordinates.

# Asymptotics

Observation:

$$K_D = (K_X \otimes [D])|_D$$
 is ample,

so there is a unique (up to a constant multiple) Kähler-Einstein metric on D.

Theorem (Schumacher '98)  $\omega_{Poin}$ , when restricted to

$$D_{\sigma_0} := \{ \sigma = \sigma_0 \},\,$$

converges to  $\omega_D$  locally uniformly as  $\sigma_0 \to 0$ .

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Poincaré-type metric

Choose local coordinates  $(\sigma, z^2, \dots, z^n)$  near a point of D. Let

- $g_{\sigma\bar{\sigma}}$  etc. be the coefficients of  $\omega_{Poin}$  and
- $ightharpoonup q^{\bar{\sigma}\sigma}$  etc. be the entries of the inverse matrix.

- $\triangleright a^{\bar{\sigma}\sigma} \sim |\sigma|^2 \log^2(1/|\sigma|^2).$
- $p = q^{\bar{\sigma}k}, q^{\bar{l}\sigma} = O(|\sigma| \log^{1-\alpha}(1/|\sigma|^2)), k, l = 2, \dots, n,$
- $ightharpoonup a^{\bar{k}k} \sim 1, \ k=2,\ldots,n$  and
- $ightharpoonup a^{\overline{l}k} \to 0$  as  $\sigma \to 0$ ,  $k, l = 2, \ldots, n, k \neq l$ .

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Choose local coordinates  $(\sigma, z^2, \dots, z^n)$  near a point of D. Let

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  ightarrow 0$  as  $\sigma 
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  eq l.

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### Let

- ► (X, D) be a canonically polarized framed manifold and
- E a holomorphic vector bundle on X.

### Questions

- How can we define a notion of "framed stability" of E?
- 2. What should a "framed Hermitian-Einstein metric" in E be?
- 3. Do we obtain existence and uniqueness of framed Hermitian-Einstein metrics in the case of framed stability?

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▶ On X: The degree with respect to the polarization  $K_X \otimes [D]$ :

$$\deg_{K_X \otimes [D]}(\mathcal{E}) = (c_1(\mathcal{E}) \cup c_1(K_X \otimes [D])^{n-1}) \cap [X]$$
$$= \int_X c_1(\mathcal{E}) \wedge \omega^{n-1},$$

where  $\omega=\frac{i}{2\pi}\cdot \text{curvature}$  form of a positive Hermitian metric in  $K_X\otimes [D].$ 

On X': The degree with respect to the Poincaré-type metric:

$$\deg_{X'}(\mathcal{E}) = \int_{X'} c_1(\mathcal{E}) \wedge \omega_{\mathsf{Poin}}^{n-1}$$

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# Theorem (S. '09)

For every torsion-free coherent sheaf  $\mathcal E$  on X, the number  $\deg_{X'}(\mathcal E)$  is well-defined and satisfies

$$\deg_{K_X \otimes [D]}(\mathcal{E}) = \deg_{X'}(\mathcal{E}).$$

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- For convenience only n=2.
- Exhaustion

$$X' = \bigcup_{\varepsilon > 0} X_{\varepsilon} \quad \text{with } X_{\varepsilon} = \{x \in X : ||\sigma(x)|| > \varepsilon\}$$

We have

$$\deg_{K_X \otimes [D]}(\mathcal{E}) = \lim_{\varepsilon \to 0} \int_{X_{\varepsilon}} c_1(\mathcal{E}) \wedge \omega,$$
$$\deg_{X'}(\mathcal{E}) = \lim_{\varepsilon \to 0} \int_{X_{\varepsilon}} c_1(\mathcal{E}) \wedge \omega_{\mathsf{Poin}}.$$

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Use the results on the asymptotics of the Poincaré-type metric. More precisely:

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#### Framed stability and Hermitian-Einstein metrics

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Framed stability

- ▶ Use asymptotic results to compare  $\omega$  and  $\omega_{Poin}$ .
- We obtain

$$\begin{split} \deg_{X'}(\mathcal{E}) &= \deg_{K_X \otimes [D]}(\mathcal{E}) \\ &- 2i \lim_{\varepsilon \to 0} \int_{X_{\varepsilon}} c_1(\mathcal{E}) \wedge \partial \bar{\partial} \log \log (1/||\sigma||^2) \\ &+ i \lim_{\varepsilon \to 0} \int_{X_{\varepsilon}} c_1(\mathcal{E}) \wedge \partial \bar{\partial} \log \left( 1 + \frac{\nu}{\log^{\alpha} (1/||\sigma||^2)} \right) \end{split}$$

- $\triangleright$  Show the vanishing of the integrals for  $\varepsilon \to 0$  using

  - ▶ that  $\nu$  is  $\mathcal{C}^{k,\beta}$  with  $k \ge 2$  in quasi-coordinates.

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Framed stability and Hermitian-Einstein metrics

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Framed stability

Let  $\mathcal{E}$  be a torsion-free coherent sheaf on X.

### Definition

We call

$$\deg_{(X,D)}(\mathcal{E}) := \deg_{K_X \otimes [D]}(\mathcal{E})$$
 
$$\stackrel{\mathsf{Th.}}{=} \deg_{X'}(\mathcal{E})$$

the framed degree of  $\mathcal{E}$ .

#### Framed stability and Hermitian-Einstein metrics

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Framed stability

### Definition

▶ If  $rank(\mathcal{E}) > 0$ , we define the framed slope of  $\mathcal{E}$  as

$$\mu_{(X,D)}(\mathcal{E}) = \frac{\deg_{(X,D)}(\mathcal{E})}{\operatorname{rank}(\mathcal{E})}.$$

 $lackbox{ We say that $\mathcal{E}$ is stable in the framed sense if}$ 

$$\mu_{(X,D)}(\mathcal{F}) < \mu_{(X,D)}(\mathcal{E})$$

holds for every coherent subsheaf  $\mathcal{F}$  of  $\mathcal{E}$  with  $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$ .

### Remark

Note that we only consider subsheaves  $\mathcal{F}$  over X rather than X'.

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## Corollary

Every holomorphic vector bundle E which is stable in the framed sense is simple, i. e.

$$\operatorname{End}(E) = \mathbb{C} \cdot \operatorname{id}_E$$
.

### Remark

The simplicity seems to hold only over X and not over X'.

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Consider Hermitian-Einstein metrics in E over X' with respect to the Poincaré-type metric.

Difficulty

For a uniqueness statement we need the simplicity of E over  $X^\prime$ , which is not given in the case of framed stability

Solution

We impose additional conditions.

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For a uniqueness statement we need the simplicity of E over  $X^\prime$ , which is not given in the case of framed stability.

### Solution

We impose additional conditions.

$$\mathcal{P} = \left\{ h \text{ Hermitian metric in } E \text{ over } X' \right.$$
 with 
$$\int_{X'} |\Lambda_{\mathsf{Poin}} F_h|_h \, dV_{\mathsf{Poin}} < \infty \right\}.$$

We know (Simpson '88):  ${\mathcal P}$  decomposes into components which are covered by charts

$$s \longmapsto he^s,$$

where s is a positive definite self-adjoint section of  $\operatorname{End}(E)$  over X' with respect to h satisfying

$$\sup_{X'} |s|_h + ||\nabla'' s||_{L^2} + ||\Delta' s||_{L^1} < \infty.$$

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One of these components contains all the metrics which extend to X. We denote this component by  $\mathcal{P}_0$ .

### Definition

By a framed Hermitian-Einstein metric in E we mean a Hermitian metric h in E over X' which satisfies

- ▶  $h \in \mathcal{P}_0$  and
- $i \Lambda_{\mathsf{Poin}} F_h = \lambda \operatorname{id}_E \text{ over } X' \text{ with } \lambda \in \mathbb{R}.$

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# Uniqueness

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Theorem (S. '09)

If E is simple and h and  $\ddot{h}$  are framed Hermitian-Einstein metrics in E, we have

$$\tilde{h} = c \cdot h$$

with a positive constant c.

Framed H-F metrics

# Uniqueness

## Proof.

- ▶ h,  $\tilde{h} \in \mathcal{P}_0$  guarantees that the framed degree of E can be computed using h or  $\tilde{h}$ .
- ► Therefore, we have

$$i\Lambda_{\mathsf{Poin}}F_h = \lambda\operatorname{id}_E = i\Lambda_{\mathsf{Poin}}F_{\tilde{h}}$$

with 
$$\lambda = \frac{2\pi\mu_{(X,D)}(E)}{(n-1)!\operatorname{vol}_{\mathsf{Poin}}(X')}$$

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# Uniqueness

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#### Framed stability and Hermitian-Einstein metrics

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Framed H-F metrics

▶ Write  $\tilde{h} = he^s$ , join h and  $\tilde{h}$  by the path  $h_t = he^{ts}$  and use Donaldson's functional

$$L(t) = \int_{X'} \int_0^t \operatorname{tr} \left( s(i\Lambda_{\mathsf{Poin}} F_{h_u} - \lambda \operatorname{id}_E) \right) du \, \frac{\omega_{\mathsf{Poin}}^n}{n!}.$$

- ▶  $h, \tilde{h} \in \mathcal{P}_0$  guarantees
  - lacktriangle well-definedness of L(t),
  - $L''(t) = ||\bar{\partial}s||_{L^2}^2,$
  - ▶ simplicity of E over X is sufficient to conclude that s is a multiple of id<sub>E</sub>.

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  - $L''(t) = ||\bar{\partial}s||_{L^2}^2,$
  - ▶ simplicity of E over X is sufficient to conclude that s is a multiple of id<sub>E</sub>.

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▶ Write  $\tilde{h} = he^s$ , join h and  $\tilde{h}$  by the path  $h_t = he^{ts}$  and use Donaldson's functional

$$L(t) = \int_{X'} \int_0^t \operatorname{tr} \left( s(i\Lambda_{\mathsf{Poin}} F_{h_u} - \lambda \operatorname{id}_E) \right) du \, \frac{\omega_{\mathsf{Poin}}^n}{n!}.$$

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# Theorem (S. '09)

If E is stable in the framed sense, there exists a unique (up to a constant multiple) framed Hermitian-Einstein metric in E.

## Proof

- ► Carry over the arguments from the classical case.
- ▶ Critical point: In the case of framed stability, one only considers subsheaves of  $\mathcal{E} = \mathcal{O}_X(E)$  over X and not over X'.
- ▶ However: In the classical proof, the destabilizing subsheaf is produced from an  $L_1^2$  section of  $\operatorname{End}(E)$ .

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Therefore, it suffices to prove the following lemma.

Lemma (S. '09)

We have

 $L_1^2(X,\operatorname{End}(E),\operatorname{\it Poincar\'e})\subset L_1^2(X,\operatorname{End}(E)).$ 

Proof

using the results on the asymptotics of the Poincaré-type metric

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Let  $(X, \mathcal{D})$  be a canonically polarized framed manifold.

## Observation

▶ For large m, (X, D) is m-framed in the sense that the  $\mathbb{Q}$ -divisor

$$K_X + \frac{m-1}{m}D$$

is ample.

### Observation

▶ (Tian-Yau '87) For such m, there exist (incomplete) Kähler-Einstein metrics  $g_m$  on X' constructed from an initial metric of the form

$$i\partial\bar{\partial}\log\left(\frac{2\Omega}{m^2||\sigma||^{2(1-1/m)}(1-||\sigma||^{2/m})^2}\right),$$

▶ whereas the Poincaré-type Kähler-Einstein metric g<sub>Poin</sub> on X' is constructed from

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# Question

Can the framed situation be seen as a "limit" of the m-framed situation as  $m \to \infty$ ?

## Problems

- ► Kobayashi-Hitchin correspondence in the *m*-framed case
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- ► Convergence of the corresponding Hermitian-Einstein metrics

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## Thank you.

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