Stability and Hermitian-Einstein metrics for vector bundles on framed manifolds

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Stability

Let (X, g) be a compact Kähler manifold of complex dimension n and $\mathcal{E}$ a torsion-free coherent sheaf on X.

Definition

- The g-degree of $\mathcal{E}$ is defined as

$$\deg_g(\mathcal{E}) = (c_1(\mathcal{E}) \cup [\omega_g]^{n-1}) \cap [X] = \int_X c_1(\mathcal{E}) \wedge \omega_g^{n-1}.$$

- If $\operatorname{rank}(\mathcal{E}) > 0$, the g-slope of $\mathcal{E}$ is defined as

$$\mu_g(\mathcal{E}) = \frac{\deg_g(\mathcal{E})}{\operatorname{rank}(\mathcal{E})}.$$

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$$\mu_g(\mathcal{E}) = \frac{\deg_g(\mathcal{E})}{\operatorname{rank}(\mathcal{E})}.$$

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Definition

- $\mathcal{E}$ is called g-semistable if

$$\mu_g(\mathcal{F}) \leqslant \mu_g(\mathcal{E})$$

holds for every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \mathrm{rank}(\mathcal{F})$.

- If, moreover,

$$\mu_g(\mathcal{F}) < \mu_g(\mathcal{E})$$

holds for every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \mathrm{rank}(\mathcal{F}) < \mathrm{rank}(\mathcal{E})$, then $\mathcal{E}$ is called g-stable.

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Definition

$\mathcal{E}$ is called g-polystable if $\mathcal{E}$ is g-semistable and $\mathcal{E}$ decomposes as a direct sum

$$\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m$$

of g-stable coherent subsheaves $\mathcal{E}_1, \ldots, \mathcal{E}_m$ of the same g-slope $\mu_g(\mathcal{E}_i) = \mu_g(\mathcal{E})$.

Remark

These notions are also defined for a holomorphic vector bundle E:
Consider $\mathcal{E} = \mathcal{O}_X(E)$.

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Stability

Lemma

Every g-stable holomorphic vector bundle E is simple, i. e.

$$\mathrm{End}(E) = \mathbb{C} \cdot \mathrm{id}_E \, .$$

Hermitian-Einstein metrics

Let E be a holomorphic vector bundle on X.

Definition

A Hermitian metric h in E is called a g-Hermitian-Einstein metric if

$$i\Lambda_g F_h = \lambda \,\mathrm{id}_E \quad \text{with } \lambda \in \mathbb{R},$$

where

- $i\Lambda_g$ = contraction with ω_g,
- F_h = Chern curvature form of (E, h).

Remark

From the Einstein equation

$$i\Lambda_g F_h = \lambda \operatorname{id}_E,$$

it follows by taking the trace and integrating, that we must have

$$\lambda = \frac{2\pi \mu_g(E)}{(n-1)! \operatorname{vol}_g(X)}.$$

Kobayashi-Hitchin correspondence

Let E be a holomorphic vector bundle.

Theorem (S. Kobayashi '82, Lübke '83)

If E admits a g-Hermitian-Einstein metric, then E is g-polystable.
In particular, if E is irreducible, then it is g-stable.

Theorem (Donaldson '83–'87, Uhlenbeck/Yau '86)

If E is g-stable, then E admits a unique (up to a constant multiple) g-Hermitian-Einstein metric.

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Proof.

- Start with a fixed "background metric" h_0.
- Construct a family (h_t) of Hermitian metrics solving the evolution equation

$$h_t^{-1}\dot{h}_t = -(i\Lambda_g F_{h_t} - \lambda\, \mathrm{id}_E)$$

  for all finite values of the time parameter t, where λ is as before.
- In case of convergence as $t \to \infty$:
  The limit is a Hermitian-Einstein metric.

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Proof.

- In case of divergence:
  - Construct a "destabilizing subsheaf" contradicting the stability hypothesis, i. e. a coherent subsheaf $\mathcal{F}$ of $\mathcal{E} = \mathcal{O}_X(E)$ with $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$ and

$$\mu_g(\mathcal{F}) \geqslant \mu_g(\mathcal{E}).$$

  - This is done by first constructing a weakly holomorphic subbundle of E, i. e. a section $\pi \in L_1^2(\operatorname{End}(E))$ satisfying

$$\pi = \pi^* = \pi^2 \quad \text{and} \quad (\mathrm{id}_E - \pi) \circ \bar{\partial}\pi = 0,$$

and applying a regularity theorem of Uhlenbeck-Yau.

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Framed manifolds

Definition

- A framed manifold or logarithmic pair is a pair (X, D) consisting of
  - a compact complex manifold X and
  - a smooth divisor D in X.
- A framed manifold (X, D) is called canonically polarized if $K_X \otimes [D]$ is ample.

Example

$(\mathbb{P}^n, V)$ is canonically polarized if $V \subset \mathbb{P}^n$ is a smooth hypersurface of degree $\geqslant n + 2$.

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Framed manifolds

Classical result:

Theorem (Yau '78)

If K_X is ample, there exists a unique (up to a constant multiple) Kähler-Einstein metric on X with negative Ricci curvature.

Poincaré-type metric

In the framed situation:

Theorem (R. Kobayashi '84)

If (X, D) is canonically polarized, there exists a unique (up to a constant multiple) complete Kähler-Einstein metric on $X' := X \setminus D$ with negative Ricci curvature.

Remark

- We call this metric the Poincaré-type metric on X'.
- Choose local coordinates $(\sigma, z^2, \dots, z^n)$ such that D is given by $\sigma = 0$. Then in these coordinates, we have

$$\omega_{\text{Poin}} \sim 2i \left(\frac{d\sigma \wedge d\bar{\sigma}}{|\sigma|^2 \log^2(1/|\sigma|^2)} + \sum_{k=2}^{n} dz^k \wedge d\bar{z}^k \right).$$

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Bounded geometry

(Cheng-Yau '80, R. Kobayashi '84, Tian-Yau '87, ...)

Definition

A local quasi-coordinate map is a holomorphic map

$$V \longrightarrow X', \quad V \subset \mathbb{C}^n \text{ open}$$

which is of maximal rank everywhere. In this case, V together with the Euclidean coordinates of $\mathbb{C}^n$ is called a local quasi-coordinate system.

Bounded geometry

Theorem

X' together with the Poincaré-type metric is of bounded geometry, i. e. there is an (infinite) family $\mathcal{V} = \{(V; v^1, \ldots, v^n)\}$ of local quasi-coordinate systems such that:

- *X' is covered by the images of the V in $\mathcal{V}$.*
- *There is an open neighbourhood U of D such that $X \setminus U$ is covered by the images of finitely many V which are coordinate systems in the ordinary sense.*
- *Every V contains an open ball of radius $\frac{1}{2}$.*
- *The coefficients of the Poincaré-type metric and their derivatives in quasi-coordinates are uniformly bounded.*

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Construction of quasi-coordinates

Choose local coordinates

$$(\Delta^n; z^1, \ldots, z^n) \text{ on } U \subset X$$

such that

$$D \cap U = \{z^1 = 0\}.$$

Then the quasi-coordinates are $(B_R(0) \times \Delta^{n-1}; v^1, \ldots, v^n)$ with $\frac{1}{2} < R < 1$ such that

$$v^1 = \frac{w^1 - a}{1 - aw^1}, \quad \text{where } z^1 = \exp\left(\frac{w^1 + 1}{w^1 - 1}\right),$$

and

$$v^i = w^i = z^i \quad \text{for } 2 \leqslant i \leqslant n,$$

where a varies over real numbers in Δ close to 1.

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Asymptotics

Theorem (Schumacher '98)

There exists $0 < \alpha \leqslant 1$ such that for all $k \in \{0, 1, \ldots\}$ and $\beta \in (0, 1)$, the volume form of the Poincaré-type metric is of the form

$$\frac{2\Omega}{||\sigma||^2 \log^2(1/||\sigma||^2)} \left(1 + \frac{\nu}{\log^\alpha(1/||\sigma||^2)}\right),$$

where

- Ω *is a smooth volume form on* X,
- σ *is a canonical section of* $[D]$, $||\cdot||$ *is a norm in* $[D]$,
- ν *lies in the Hölder space of* $\mathcal{C}^{k,\beta}$ *functions in quasi-coordinates.*

Asymptotics

Observation:

$$K_D = (K_X \otimes [D])|_D \quad \text{is ample,}$$

so there is a unique (up to a constant multiple) Kähler-Einstein metric on D.

Theorem (Schumacher '98)

ω_{Poin}, when restricted to

$$D_{\sigma_0} := \{\sigma = \sigma_0\},$$

converges to ω_D locally uniformly as $\sigma_0 \to 0$.

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Asymptotics

Choose local coordinates $(\sigma, z^2, \ldots, z^n)$ near a point of D. Let

- $g_{\sigma\bar\sigma}$ etc. be the coefficients of ω_{Poin} and
- $g^{\bar\sigma\sigma}$ etc. be the entries of the inverse matrix.

Theorem (Schumacher '02)

- $g^{\bar\sigma\sigma} \sim |\sigma|^2 \log^2(1/|\sigma|^2)$,
- $g^{\bar\sigma k}, g^{\bar l\sigma} = O\big(|\sigma| \log^{1-\alpha}(1/|\sigma|^2)\big)$, $k, l = 2, \ldots, n$,
- $g^{\bar k k} \sim 1$, $k = 2, \ldots, n$ *and*
- $g^{\bar l k} \to 0$ *as* $\sigma \to 0$, $k, l = 2, \ldots, n$, $k \neq l$.

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Let

- (X, D) be a canonically polarized framed manifold and
- E a holomorphic vector bundle on X.

Questions

1. How can we define a notion of "framed stability" of E?
2. What should a "framed Hermitian-Einstein metric" in E be?
3. Do we obtain existence and uniqueness of framed Hermitian-Einstein metrics in the case of framed stability?

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Framed stability

Two notions of degree for a torsion-free coherent sheaf $\mathcal{E}$:

- On X: The degree with respect to the polarization $K_X \otimes [D]$:

  $$\deg_{K_X \otimes [D]}(\mathcal{E}) = (c_1(\mathcal{E}) \cup c_1(K_X \otimes [D])^{n-1}) \cap [X]$$
  $$= \int_X c_1(\mathcal{E}) \wedge \omega^{n-1},$$

  where $\omega = \frac{i}{2\pi} \cdot$curvature form of a positive Hermitian metric in $K_X \otimes [D]$.
- On X': The degree with respect to the Poincaré-type metric:

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- On X': The degree with respect to the Poincaré-type metric:

$$\deg_{X'}(\mathcal{E}) = \int_{X'} c_1(\mathcal{E}) \wedge \omega_{\text{Poin}}^{n-1}.$$

Framed stability

Theorem (S. '09)

For every torsion-free coherent sheaf $\mathcal{E}$ *on* X, *the number* $\deg_{X'}(\mathcal{E})$ *is well-defined and satisfies*

$$\deg_{K_X \otimes [D]}(\mathcal{E}) = \deg_{X'}(\mathcal{E}).$$

Framed stability

Proof.

Use the results on the asymptotics of the Poincaré-type metric. More precisely:

- For convenience only $n = 2$.
- Exhaustion

$$X' = \bigcup_{\varepsilon > 0} X_\varepsilon \quad \text{with } X_\varepsilon = \{x \in X : ||\sigma(x)|| > \varepsilon\}.$$

- We have

$$\deg_{K_X \otimes [D]}(\mathcal{E}) = \lim_{\varepsilon \to 0} \int_{X_\varepsilon} c_1(\mathcal{E}) \wedge \omega,$$

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Framed stability

Proof.

- Use asymptotic results to compare ω and ω_{Poin}.
- We obtain

$$\deg_{X'}(\mathcal{E}) = \deg_{K_X \otimes [D]}(\mathcal{E}) - 2i \lim_{\varepsilon \to 0} \int_{X_\varepsilon} c_1(\mathcal{E}) \wedge \partial\bar{\partial} \log\log(1/||\sigma||^2) + i \lim_{\varepsilon \to 0} \int_{X_\varepsilon} c_1(\mathcal{E}) \wedge \partial\bar{\partial} \log\left(1 + \frac{\nu}{\log^\alpha(1/||\sigma||^2)}\right)$$

- Show the vanishing of the integrals for $\varepsilon \to 0$ using
  - Stokes' theorem and
  - that ν is $\mathcal{C}^{k,\beta}$ with $k \geqslant 2$ in quasi-coordinates.

Framed stability

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Framed stability

Let $\mathcal{E}$ be a torsion-free coherent sheaf on X.

Definition

We call

$$\deg_{(X,D)}(\mathcal{E}) := \deg_{K_X \otimes [D]}(\mathcal{E}) \overset{\text{Th.}}{=} \deg_{X'}(\mathcal{E})$$

the framed degree of $\mathcal{E}$.

Framed stability

Definition

- If $\mathrm{rank}(\mathcal{E}) > 0$, we define the framed slope of $\mathcal{E}$ as

$$\mu_{(X,D)}(\mathcal{E}) = \frac{\deg_{(X,D)}(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}.$$

- We say that $\mathcal{E}$ is stable in the framed sense if

$$\mu_{(X,D)}(\mathcal{F}) < \mu_{(X,D)}(\mathcal{E})$$

holds for every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \mathrm{rank}(\mathcal{F}) < \mathrm{rank}(\mathcal{E})$.

Remark

Note that we only consider subsheaves $\mathcal{F}$ over X rather than X'.

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Corollary

Every holomorphic vector bundle E which is stable in the framed sense is simple, i. e.

$$\mathrm{End}(E) = \mathbb{C} \cdot \mathrm{id}_E \, .$$

Remark

The simplicity seems to hold only over X and not over X'.

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Framed Hermitian-Einstein metrics

Consider Hermitian-Einstein metrics in E over X' with respect to the Poincaré-type metric.

Difficulty

For a uniqueness statement we need the simplicity of E over X', which is not given in the case of framed stability.

Solution

We impose additional conditions.

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Framed Hermitian-Einstein metrics

Set

$$\mathcal{P} = \Big\{ h \text{ Hermitian metric in } E \text{ over } X' \text{ with } \int_{X'} |\Lambda_{\mathsf{Poin}} F_h|_h \, dV_{\mathsf{Poin}} < \infty \Big\}.$$

We know (Simpson '88): $\mathcal{P}$ decomposes into components which are covered by charts

$$s \longmapsto he^s,$$

where s is a positive definite self-adjoint section of $\mathrm{End}(E)$ over X' with respect to h satisfying

$$\sup_{X'} |s|_h + ||\nabla'' s||_{L^2} + ||\Delta' s||_{L^1} < \infty.$$

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Framed Hermitian-Einstein metrics

One of these components contains all the metrics which extend to X. We denote this component by $\mathcal{P}_0$.

Definition

By a framed Hermitian-Einstein metric in E we mean a Hermitian metric h in E over X' which satisfies

- $h \in \mathcal{P}_0$ and
- $i\Lambda_{\mathsf{Poin}} F_h = \lambda \, \mathrm{id}_E$ over X' with $\lambda \in \mathbb{R}$.

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Uniqueness

Theorem (S. '09)

If E is simple and h and $\tilde{h}$ are framed Hermitian-Einstein metrics in E, we have

$$\tilde{h} = c \cdot h$$

with a positive constant c.

Uniqueness

Proof.

- h, $\tilde{h} \in \mathcal{P}_0$ guarantees that the framed degree of E can be computed using h or $\tilde{h}$.
- Therefore, we have

$$i\Lambda_{\mathsf{Poin}} F_h = \lambda \, \mathrm{id}_E = i\Lambda_{\mathsf{Poin}} F_{\tilde{h}}$$

$$\text{with} \quad \lambda = \frac{2\pi\mu_{(X,D)}(E)}{(n-1)! \, \mathrm{vol}_{\mathsf{Poin}}(X')}.$$

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Uniqueness

Proof.

- Write $\tilde{h} = he^s$, join h and $\tilde{h}$ by the path $h_t = he^{ts}$ and use Donaldson's functional

$$L(t) = \int_{X'} \int_0^t \operatorname{tr}\left(s(i\Lambda_{\mathsf{Poin}} F_{h_u} - \lambda \operatorname{id}_E)\right) du \, \frac{\omega_{\mathsf{Poin}}^n}{n!}.$$

- h, $\tilde{h} \in \mathcal{P}_0$ guarantees
  - well-definedness of $L(t)$,
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Existence in the case of framed stability

Theorem (S. '09)

If E is stable in the framed sense, there exists a unique (up to a constant multiple) framed Hermitian-Einstein metric in E.

Proof.

- Carry over the arguments from the classical case.
- Critical point: In the case of framed stability, one only considers subsheaves of $\mathcal{E} = \mathcal{O}_X(E)$ over X and not over X'.
- However: In the classical proof, the destabilizing subsheaf is produced from an L_1^2 section of $\mathrm{End}(E)$.

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Existence in the case of framed stability

Therefore, it suffices to prove the following lemma.

Lemma (S. '09)

We have

$$L^2_1(X, \mathrm{End}(E), \textit{Poincaré}) \subset L^2_1(X, \mathrm{End}(E)).$$

Proof.

using the results on the asymptotics of the Poincaré-type metric

Existence in the case of framed stability

Therefore, it suffices to prove the following lemma.

Lemma (S. '09)

We have

$$L_1^2(X, \mathrm{End}(E), \textit{Poincaré}) \subset L_1^2(X, \mathrm{End}(E)).$$

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Contents

Outlook

Let (X, D) be a canonically polarized framed manifold.

Observation

- For large m, (X, D) is m-framed in the sense that the $\mathbb{Q}$-divisor
$$K_X + \frac{m-1}{m} D$$
is ample.

Outlook

Observation

- (Tian-Yau '87) For such m, there exist (incomplete) Kähler-Einstein metrics g_m on X' constructed from an initial metric of the form

$$i\partial\bar{\partial}\log\left(\frac{2\Omega}{m^2||\sigma||^{2(1-1/m)}(1-||\sigma||^{2/m})^2}\right),$$

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Can the framed situation be seen as a "limit" of the m-framed situation as $m \to \infty$?

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- Kobayashi-Hitchin correspondence in the m-framed case
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Thank you.