Stability and Hermitian-Einstein metrics for vector bundles on framed manifolds

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Lie Theory and Complex Geometry Marburg, November 4, 2010

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Let (X,g) be a compact Kähler manifold of complex dimension n and \mathcal{E} a torsion-free coherent sheaf on X.

Definition

lacktriangle The g-degree of ${\mathcal E}$ is defined as

$$\deg_g(\mathcal{E}) = \int_X c_1(\mathcal{E}) \wedge \omega_g^{n-1}$$

▶ If $rank(\mathcal{E}) > 0$, the *g*-slope of \mathcal{E} is defined as

$$\mu_g(\mathcal{E}) = \frac{\deg_g(\mathcal{E})}{\operatorname{rank}(\mathcal{E})}$$

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Definition

 ${\mathcal E}$ is called $g ext{-stable}$ if

$$\mu_g(\mathcal{F}) < \mu_g(\mathcal{E})$$

holds for every coherent subsheaf $\mathcal F$ of $\mathcal E$ with $0<\mathrm{rank}(\mathcal F)<\mathrm{rank}(\mathcal E).$

Remark

This is also defined for a holomorphic vector bundle E: Consider $\mathcal{E} = \mathcal{O}_X(E)$. Framed stability and Hermitian-Einstein metrics

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Hermitian-Einstein metrics

Let E be a holomorphic vector bundle on X.

Definition

A Hermitian metric h in E is called a q-Hermitian-Einstein metric if

$$i\Lambda_g F_h = \lambda \operatorname{id}_E \quad \text{with } \lambda \in \mathbb{R},$$

where

- $i\Lambda_q = \text{contraction with } \omega_q$,
- $ightharpoonup F_h = \text{Chern curvature form of } (E, h).$

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Theorem (S. Kobayashi '82, Lübke '83)

E is irreducible and admits a g-Hermitian-Einstein metric $\Longrightarrow E$ is g-stable.

Theorem (Donaldson '83–'87, Uhlenbeck/Yau '86) E is g-stable $\Longrightarrow E$ admits a unique (up to a constant multiple) g-Hermitian-Einstein metric.

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Proof.

- ▶ Start with a fixed "background metric" h_0 .
- ▶ Solve an evolution equation for all finite values of a time parameter t.
- ▶ In case of convergence as $t \to \infty$: The limit is a Hermitian-Einstein metric.
- ► In case of divergence: Construct a "destabilizing subsheaf" contradicting the stability hypothesis.

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- A framed manifold or logarithmic pair is a pair (X, D) consisting of
 - ightharpoonup a compact complex manifold X and
 - ightharpoonup a smooth divisor D in X.
- ▶ A framed manifold (X, D) is called canonically polarized if $K_X \otimes [D]$ is ample.

Example

 (\mathbb{P}^n,V) is canonically polarized if $V\subset\mathbb{P}^n$ is a smooth hypersurface of degree $\geqslant n+2$.

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- lacktriangle We call this metric the Poincaré-type metric on X'.
- ▶ Choose local coordinates $(\sigma, z^2, \dots, z^n)$ such that D is given by $\sigma = 0$. Then in these coordinates, we have

$$\omega_{\mathsf{Poin}} \sim 2i \left(\frac{d\sigma \wedge d\bar{\sigma}}{|\sigma|^2 \log^2(1/|\sigma|^2)} + \sum_{k=2}^n dz^k \wedge d\bar{z}^k \right).$$

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(Cheng-Yau '80, R. Kobayashi '84, Tian-Yau '87, ...)

Definition

A local quasi-coordinate map is a holomorphic map

$$V \longrightarrow X', \quad V \subset \mathbb{C}^n$$
 open

which is of maximal rank everywhere. In this case, V together with the Euclidean coordinates of \mathbb{C}^n is called a local quasi-coordinate system.

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Theorem

X' together with the Poincaré-type metric is of bounded geometry, i. e. there is an (infinite) family $\mathcal{V} = \{(V; v^1, \dots, v^n)\}$ of local quasi-coordinate systems such that:

- ightharpoonup X' is covered by the images of the V in $\mathcal V$.
- ▶ There is an open neighbourhood U of D such that $X \setminus U$ is covered by the images of finitely many V which are coordinate systems in the ordinary sense.
- ▶ Every V contains an open ball of radius $\frac{1}{2}$.
- ► The coefficients of the Poincaré-type metric and their derivatives in quasi-coordinates are uniformly bounded.

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$$(\Delta^n;z^1,\dots,z^n)$$
 on $U\subset X$

such that

$$D \cap U = \{ z^1 = 0 \}.$$

Then the quasi-coordinates are $(P_n(0) \times A^{n-1}, \dots, 1 \times P_n)$ with

$$(B_R(0) \times \Delta^{n-1}; v^1, \dots, v^n)$$
 with $\frac{1}{2} < R < 1$ such that

$$v^{1} = \frac{w^{1} - a}{1 - aw^{1}}, \text{ where } z^{1} = \exp\left(\frac{w^{1} + 1}{w^{1} - 1}\right).$$

and

$$v^i = w^i = z^i$$
 for $2 \leqslant i \leqslant n$,

where a varies over real numbers in Δ close to 1.

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Theorem (Schumacher '98)

There exists $0 < \alpha \le 1$ such that for all $k \in \{0, 1, \ldots\}$ and $\beta \in (0, 1)$, the volume form of the Poincaré-type metric is of the form

$$\frac{2\Omega}{||\sigma||^2\log^2(1/||\sigma||^2)}\left(1+\frac{\nu}{\log^\alpha(1/||\sigma||^2)}\right),$$

where

- $ightharpoonup \Omega$ is a smooth volume form on X,
- $ightharpoonup \sigma$ is a canonical section of [D], $||\cdot||$ is a norm in [D],
- $\blacktriangleright \nu$ lies in the Hölder space of $\mathcal{C}^{k,\beta}$ functions in quasi-coordinates.

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Let

- ► (X, D) be a canonically polarized framed manifold and
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Questions

- How can we define a notion of "framed stability" of E?
- 2. What should a "framed Hermitian-Einstein metric" in E be?
- 3. Do we obtain existence and uniqueness of framed Hermitian-Einstein metrics in the case of framed stability?

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▶ On X: The degree with respect to the polarization $K_X \otimes [D]$:

$$\deg_{K_X \otimes [D]}(\mathcal{E}) = (c_1(\mathcal{E}) \cup c_1(K_X \otimes [D])^{n-1}) \cap [X]$$
$$= \int_X c_1(\mathcal{E}) \wedge \omega^{n-1},$$

where $\omega=\frac{i}{2\pi}\cdot \text{curvature}$ form of a positive Hermitian metric in $K_X\otimes [D].$

➤ On X': The degree with respect to the Poincaré-type metric:

$$\deg_{X'}(\mathcal{E}) = \int_{X'} c_1(\mathcal{E}) \wedge \omega_{\mathsf{Poin}}^{n-1}$$

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Theorem (S. '09)

For every torsion-free coherent sheaf $\mathcal E$ on X, the number $\deg_{X'}(\mathcal E)$ is well-defined and satisfies

$$\deg_{K_X \otimes [D]}(\mathcal{E}) = \deg_{X'}(\mathcal{E}).$$

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Relationship

- For convenience only n=2.
- Exhaustion

$$X' = \bigcup_{\varepsilon > 0} X_{\varepsilon} \quad \text{with } X_{\varepsilon} = \{x \in X : ||\sigma(x)|| > \varepsilon\}$$

We have

$$\deg_{K_X \otimes [D]}(\mathcal{E}) = \lim_{\varepsilon \to 0} \int_{X_{\varepsilon}} c_1(\mathcal{E}) \wedge \omega,$$
$$\deg_{X'}(\mathcal{E}) = \lim_{\varepsilon \to 0} \int_{X_{\varepsilon}} c_1(\mathcal{E}) \wedge \omega_{\mathsf{Poin}}$$

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Relationship

- ▶ Use asymptotic results to compare ω and ω_{Poin} .
- ► We obtain

$$\begin{split} \deg_{X'}(\mathcal{E}) &= \deg_{K_X \otimes [D]}(\mathcal{E}) \\ &- 2i \lim_{\varepsilon \to 0} \int_{X_{\varepsilon}} c_1(\mathcal{E}) \wedge \partial \bar{\partial} \log \log (1/||\sigma||^2) \\ &+ i \lim_{\varepsilon \to 0} \int_{X_{\varepsilon}} c_1(\mathcal{E}) \wedge \partial \bar{\partial} \log \left(1 + \frac{\nu}{\log^{\alpha} (1/||\sigma||^2)}\right) \end{split}$$

- ightharpoonup Show the vanishing of the integrals for arepsilon o 0 using
 - Stokes' theorem and
 - ▶ that ν is $\mathcal{C}^{k,\beta}$ with $k \ge 2$ in quasi-coordinates.

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Framed stability

Let \mathcal{E} be a torsion-free coherent sheaf on X.

Definition

We call

$$\deg_{(X,D)}(\mathcal{E}) := \deg_{K_X \otimes [D]}(\mathcal{E})$$

$$\stackrel{\mathsf{Th.}}{=} \deg_{X'}(\mathcal{E})$$

the framed degree of \mathcal{E} .

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Definition

▶ If $rank(\mathcal{E}) > 0$, we define the framed slope of \mathcal{E} as

$$\mu_{(X,D)}(\mathcal{E}) = \frac{\deg_{(X,D)}(\mathcal{E})}{\operatorname{rank}(\mathcal{E})}.$$

 $lackbox{ We say that \mathcal{E} is stable in the framed sense if}$

$$\mu_{(X,D)}(\mathcal{F}) < \mu_{(X,D)}(\mathcal{E})$$

holds for every coherent subsheaf \mathcal{F} of \mathcal{E} with $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$.

Remark

Note that we only consider subsheaves \mathcal{F} over X rather than X'.

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Corollary

Every holomorphic vector bundle E which is stable in the framed sense is simple, i. e.

$$\operatorname{End}(E) = \mathbb{C} \cdot \operatorname{id}_E$$
.

Remark

The simplicity seems to hold only over X and not over X'.

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Dutlook

Consider Hermitian-Einstein metrics in E over X' with respect to the Poincaré-type metric.

Difficulty

For a uniqueness statement we need the simplicity of E over X', which is not given in the case of framed stability

Solution

We impose additional conditions.

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Dutlook

Consider Hermitian-Einstein metrics in E over X' with respect to the Poincaré-type metric.

Difficulty

For a uniqueness statement we need the simplicity of E over X^\prime , which is not given in the case of framed stability.

Solution

We impose additional conditions.

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)utlook

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Difficulty

For a uniqueness statement we need the simplicity of E over X^\prime , which is not given in the case of framed stability.

Solution

We impose additional conditions.

$$\mathcal{P} = \left\{ h \text{ Hermitian metric in } E \text{ over } X' \right.$$
 with
$$\int_{X'} |\Lambda_{\mathsf{Poin}} F_h|_h \, dV_{\mathsf{Poin}} < \infty \right\}.$$

We know (Simpson '88): ${\cal P}$ decomposes into components which are covered by charts

$$s \longmapsto he^s,$$

where s is a positive definite self-adjoint section of $\operatorname{End}(E)$ over X' with respect to h satisfying

$$\sup_{Y'} |s|_h + ||\nabla'' s||_{L^2} + ||\Delta' s||_{L^1} < \infty.$$

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Jutlook

One of these components contains all the metrics which extend to X. We denote this component by \mathcal{P}_0 .

Definition

By a framed Hermitian-Einstein metric in E we mean a Hermitian metric h in E over X' which satisfies

- ▶ $h \in \mathcal{P}_0$ and
- $i \Lambda_{\mathsf{Poin}} F_h = \lambda \operatorname{id}_E \text{ over } X' \text{ with } \lambda \in \mathbb{R}.$

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Uniqueness

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Theorem (S. '09)

If E is simple and h and \tilde{h} are framed Hermitian-Einstein metrics in E, we have

$$\tilde{h} = c \cdot h$$

with a positive constant c.

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Uniqueness

Proof.

- ▶ h, $\tilde{h} \in \mathcal{P}_0$ guarantees that the framed degree of E can be computed using h or \tilde{h} .
- ► Therefore, we have

$$i\Lambda_{\mathsf{Poin}}F_h = \lambda\operatorname{id}_E = i\Lambda_{\mathsf{Poin}}F_{\tilde{h}}$$

with
$$\lambda = \frac{2\pi\mu_{(X,D)}(E)}{(n-1)!\operatorname{vol}_{\mathsf{Poin}}(X')}$$

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$$L(t) = \int_{X'} \int_0^t \operatorname{tr} \left(s(i\Lambda_{\mathsf{Poin}} F_{h_u} - \lambda \operatorname{id}_E) \right) du \, \frac{\omega_{\mathsf{Poin}}^n}{n!}.$$

- ▶ $h, \tilde{h} \in \mathcal{P}_0$ guarantees
 - lacktriangle well-definedness of L(t),
 - $L''(t) = ||\bar{\partial}s||_{L^2}^2,$
 - ▶ simplicity of E over X is sufficient to conclude that s is a multiple of id_E.

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Existence in the case of framed stability

Theorem (S. '09)

If E is stable in the framed sense, there exists a unique (up to a constant multiple) framed Hermitian-Einstein metric in E.

Proof

- ► Carry over the arguments from the classical case.
- ▶ Critical point: In the case of framed stability, one only considers subsheaves of $\mathcal{E} = \mathcal{O}_X(E)$ over X and not over X'.
- ▶ However: In the classical proof, the destabilizing subsheaf is produced from an L_1^2 section of $\operatorname{End}(E)$.

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Therefore, it suffices to prove the following lemma.

Lemma (S. '09)

We have

 $L_1^2(X,\operatorname{End}(E),\operatorname{\it Poincar\'e})\subset L_1^2(X,\operatorname{End}(E)).$

Proof

using the results on the asymptotics of the Poincaré-type metric

Existence in the case of framed stability

Therefore, it suffices to prove the following lemma.

Lemma (S. '09)

We have

$$L_1^2(X,\operatorname{End}(E),\operatorname{\textit{Poincar\'e}})\subset L_1^2(X,\operatorname{End}(E)).$$

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Outlook

Let (X,\mathcal{D}) be a canonically polarized framed manifold.

Observation

▶ For large m, (X, D) is m-framed in the sense that the \mathbb{Q} -divisor

$$K_X + \frac{m-1}{m}D$$

is ample.

Observation

 \blacktriangleright (Tian-Yau '87) For such m, there exist (incomplete) Kähler-Einstein metrics q_m on X' constructed from an initial metric of the form

$$i\partial\bar{\partial}\log\left(\frac{2\Omega}{m^2||\sigma||^{2(1-1/m)}(1-||\sigma||^{2/m})^2}\right),$$

whereas the Poincaré-type Kähler-Einstein metric

$$i\partial\bar{\partial}\log\left(\frac{2\Omega}{||\sigma||^2\log^2(1/||\sigma||^2)}\right)$$

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Observation

► (Tian-Yau '87) For such m, there exist (incomplete) Kähler-Einstein metrics g_m on X' constructed from an initial metric of the form

$$i\partial\bar{\partial}\log\left(\frac{2\Omega}{m^2||\sigma||^{2(1-1/m)}(1-||\sigma||^{2/m})^2}\right),$$

▶ whereas the Poincaré-type Kähler-Einstein metric g_{Poin} on X' is constructed from

$$i\partial\bar{\partial}\log\left(\frac{2\Omega}{||\sigma||^2\log^2(1/||\sigma||^2)}\right).$$

Question

Can the framed situation be seen as a "limit" of the m-framed situation as $m \to \infty$?

Problems

- ► Kobayashi-Hitchin correspondence in the *m*-framed case
- ightharpoonup Convergence of g_m to g_{Poin}
- ► Convergence of the corresponding Hermitian-Einstein metrics

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Adapting the notions

Problem Framed stability Framed H-E metrics Relationship



Thank you.

Framed stability and Hermitian-Einstein metrics

Matthias Stemmler

Introduction

Stability

Hermitian-Einstein metrics

H correspondence

Framed mar

Poincaré-type metric

Adapting the notions

Problem

Framed stability Framed H-E metric