Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER Stochastic Processes and their Applications 115 (2005) 1518–1529
stochastic processes and their applications
www.elsevier.com/locate/spa

Martingale approximations for continuous-time and discrete-time stationary Markov processes

Hajo Holzmann*,[1]

Institut für Mathematische Stochastik, Georg-August-Universität Göttingen, Maschmühlenweg 8-10, D-37073 Göttingen, Germany

Received 1 July 2004; received in revised form 14 March 2005; accepted 1 April 2005
Available online 27 April 2005

Abstract

We show that the method of Kipnis and Varadhan [Comm. Math. Phys. 104 (1986) 1–19] to construct a Martingale approximation to an additive functional of a stationary ergodic Markov process via the resolvent is universal in the sense that a martingale approximation exists if and only if the resolvent representation converges. A sufficient condition for the existence of a martingale approximation is also given. As examples we discuss moving average processes and processes with normal generator.

MSC: primary 60F05; 60G44; secondary 60J25; 60J35

Keywords: Markov process; Central limit theorem; Martingale; Moving average process; Normal operator

1. Introduction

The central limit theorem (CLT) for additive functionals of stationary, ergodic Markov chains has been studied intensively during the last decades. A basic approach for proving the CLT, initiated by Gordin and Lifšic [13] and afterwards

*Tel.: +49 551 3913514; fax: +49 551 13505.
E-mail address: holzmann@math.uni-goettingen.de.
[1]The author acknowledges financial support of the graduate school 'Gruppen und Geometrie'.

0304-4149/$ - see front matter
doi:10.1016/j.spa.2005.04.001

pursued by several authors, is to construct a martingale approximation to the partial sums. These are decomposed into a sum of a martingale with stationary increments and a remainder term. After showing that the remainder term is negligible in some suitable sense, asymptotic normality follows from a martingale CLT. In this note we will focus on the case where the remainder is negligible in mean-square. Let $(X_n)_{n\geqslant 0}$ be a stationary ergodic (discrete-time) Markov chain with state space $(X, \mathscr{B})$, transition operator Q and stationary initial distribution μ. We denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the norm and the inner product of $L_2(\mu)$, respectively, and by $L_2^0(\mu)$ the subspace with $\int f\,\mathrm{d}\mu = 0$. For a fixed function $f \in L_2^0(\mu)$ let $S_0 = 0$ and

$$S_n(f) = f(X_1) + \cdots + f(X_n), \quad n \geqslant 1.$$

Definition 1.1. We say that there is a *martingale approximation* to $S_n(f)$ if there exist two sequences of random variables $(M_n)_{n\geqslant 1}$ and $(A_n)_{n\geqslant 1}$ such that

1. $S_n(f) = M_n + A_n, \quad n \geqslant 1$,
2. $(M_n)_{n\geqslant 1}$ is a square-integrable martingale with stationary increments with respect to $\mathscr{F}_n = \sigma(X_0, \ldots, X_n)$,
3. $E(A_n)^2/n \to 0,\ n \to \infty$.

Notice that if there exists a martingale approximation to $S_n(f)$, the processes $(M_n)_{n\geqslant 1}$ and $(A_n)_{n\geqslant 1}$ are uniquely determined a.s.. Given a martingale approximation to $S_n(f)$, from the CLT for martingales with stationary, ergodic increments due to Billingsley [2] and Ibragimov [16] it follows that:

$$\frac{S_n(f)}{\sqrt{n}} \overset{n\to\infty}{\Rightarrow} \mathrm{N}(0, \sigma^2(f)),$$

where the asymptotic variance satisfies

$$\sigma^2(f) = EM_1^2 = \lim_{n\to\infty} ES_n(f)^2/n.$$

A martingale approximation in the above sense (with additional properties in several cases) was constructed by Derriennic and Lin [9–11], Gordin and Holzmann [12], Gordin and Lifšic [13,14], Kipnis and Varadhan [17], Maxwell and Woodroofe [18] and Woodroofe [20] under suitable conditions on the function f and in some cases on the Markov operator Q. Wu and Woodroofe [21] also investigated necessary and sufficient conditions for existence of (a different notion of) martingale approximations. For a comparison with their results see Remark 2.4.

In this note we will mainly consider continuous-time Markov processes. Let $(X_t)_{t\geqslant 0}$ be a stationary ergodic Markov process, defined on a probability space $(\Omega, \mathscr{A}, P)$, with state space $(X, \mathscr{B})$, transition probability function $p(t, x, \mathrm{d}y)$ and stationary initial distribution μ. We assume that the contraction semigroup

$$T_t f(x) = \int_X f(y)\, p(t, x, \mathrm{d}y), \quad f \in L_2(\mu)$$

is strongly continuous (on $L_2(\mu)$). Let $(\mathscr{F}_t)_{t\geqslant 0}$ be a filtration in $(\Omega, \mathscr{A}, P)$ such that $(X_t)_{t\geqslant 0}$ is progressively measurable with respect to $(\mathscr{F}_t)_{t\geqslant 0}$ and satisfies the

Markov property

$$E(f(X_t)|\mathscr{F}_u) = T_{t-u}f(X_u), \quad f \in L_2(\mu), \quad 0 \leqslant u < t. \tag{1}$$

Let L be the generator of $(T_t)_{t\geqslant 0}$ and $\mathscr{D}(L)$ its domain of definition on $L_2(\mu)$. Denote by $L_2^0(\mu)$ the subspace of functions in $L_2(\mu)$ with $\int_X f \, d\mu = 0$. For $f \in L_2^0(\mu)$ and $t \geqslant 0$ let

$$S_t(f) = \int_0^t f(X_s) \, ds.$$

For a more detailed description of this setting see e.g. [1].

Definition 1.2. We say that there is a *martingale approximation* to $S_t(f)$ if there exist two processes $(M_t)_{t\geqslant 0}$ and $(A_t)_{t\geqslant 0}$ on $(\Omega, \mathscr{A}, P)$ such that

1. $S_t(f) = M_t + A_t, \quad t \geqslant 0,$
2. $(M_t)_{t\geqslant 0}$ is a square-integrable martingale with respect to $(\mathscr{F}_t)_{t\geqslant 0}$ with stationary increments and $M_0 = 0$,
3. $E(A_t)^2/t \to 0$ as $t \to \infty$.

Again note that once the filtration $(\mathscr{F}_t)_{t\geqslant 0}$ is fixed, a martingale approximation is uniquely determined a.s.. As in the discrete-time case, using a CLT for martingales with stationary increments (a careful discussion of which can be found in [6]) the existence of a martingale approximation implies that

$$\frac{S_t(f)}{\sqrt{t}} \overset{t\to\infty}{\Rightarrow} \mathrm{N}(0, \sigma^2(f)),$$

where

$$\sigma^2(f) = EM_1^2 = \lim_{t\to\infty} ES_t(f)^2/t.$$

The problem of the validity of the CLT for general continuous-time Markov processes has been studied less intensively than for discrete-time chains, and there seem to be few results via martingale approximation. Bhattacharya [1] proved the continuous-time analogue of Gordin and Lifšic's [13] result. He assumed that there exists a solution to Poisson's equation

$$f = -Lg, \quad g \in \mathscr{D}(L). \tag{2}$$

Write

$$S_t(f) = g(X_t) - g(X_0) - \int_0^t Lg(X_s) \, ds + g(X_0) - g(X_t).$$

Using Dynkin's formula

$$T_t g - g = \int_0^t T_s Lg \, ds, \quad g \in \mathscr{D}(L) \tag{3}$$

it can be shown that $(M_t)_{t\geqslant 0}$, $M_t = g(X_t) - g(X_0) - \int_0^t Lg(X_s) \, ds$, is a martingale with stationary increments with respect to $(\mathscr{F}_t)$. Furthermore, we evidently have

$E(g(X_0)-g(X_t))^2/t \to 0$. For the asymptotic variance, Bhattacharya [1] gave the formula

$$\sigma^2(f) = 2\langle f, g\rangle \text{ where } f = -Lg.$$

Kipnis and Varadhan [17] extended this approach in the context of reversible processes by solving (2) approximately via the resolvent. Using their method, de Masi et al. [8] studied reversible processes under weaker integrability conditions on the function f, and Olla [19] considered non-reversible processes via the symmetrized operator $(L+L^*)/2$. Here our main goal is to show that the method of Kipnis and Varadhan [17] is universal in a certain sense. In Section 2 we introduce the resolvent representation of $S_t(f)$ and show that there exists a martingale approximation to $S_t(f)$ if and only if the resolvent representation converges (for the definitions see Section 2). Corresponding results are also formulated for Markov chains. In Section 3 this is applied to prove the CLT for stationary Markov processes under a condition analogous to that used by Maxwell and Woodroofe [18] in the discrete-time setting. As an example we consider moving average processes in continuous time. Furthermore, we give a sufficient condition for the existence of a martingale approximation if the generator L is a normal operator on $L_2^{\mathbb{C}}(\mu)$.

2. Martingale approximation and the resolvent

Let us start this section by recalling the resolvent representation of $S_t(f)$, as introduced in [17]. Given $\varepsilon > 0$ let

$$R_\varepsilon f = (\varepsilon I - L)^{-1} f = \int_0^\infty \mathrm{e}^{-\varepsilon t} T_t f \,\mathrm{d}t, \quad f \in L_2(\mu),$$

be the resolvent. Since $R_\varepsilon f \in \mathscr{D}(L)$, given $f \in L_2^0(\mu)$ we let $g_\varepsilon = R_\varepsilon f$ and decompose

$$S_t(f) = M_{t,\varepsilon} + \varepsilon S_t(g_\varepsilon) + A_{t,\varepsilon}, \tag{4}$$

where

$$M_{t,\varepsilon} = g_\varepsilon(X_t) - g_\varepsilon(X_0) - \int_0^t (Lg_\varepsilon)(X_s)\,\mathrm{d}s, \quad A_{t,\varepsilon} = -g_\varepsilon(X_t) + g_\varepsilon(X_0).$$

For each $\varepsilon > 0$ the process $(M_{t,\varepsilon})_{t \geqslant 0}$ is a square-integrable martingale with stationary increments and $M_{0,\varepsilon} = 0$.

Definition 2.1. The decomposition (4) of $S_t(f)$ is called the *resolvent representation*. The resolvent representation is said to *converge* if

1. $\varepsilon \|g_\varepsilon\|^2 \to 0, \ \varepsilon \to 0$.
2. There exists a decreasing sequence $\varepsilon_n \to 0$ with $\varepsilon_{n+1} \geqslant c\, \varepsilon_n$ for some $c > 0$ such that for each $t \geqslant 0$, M_{t,ε_n} converges as $n \to \infty$ to a limit in $L_2(\Omega, \mathscr{A}, P)$.

Although this definition is rather technical, its significance becomes clear in the following theorem.

Theorem 2.2. *Let $(X_t)_{t\geqslant 0}$ be a progressively measurable stationary ergodic Markov process, defined on a probability space $(\Omega, \mathscr{A}, P)$, with state space $(X, \mathscr{B})$, strongly continuous contraction semigroup $(T_t)_{t\geqslant 0}$ and stationary initial distribution μ. Let $f \in L_2^0(\mu)$ and $S_t(f) = \int_0^t f(X_s)\,\mathrm{d}s$. Then there exists a martingale approximation to $S_t(f)$ if and only if the resolvent representation of $S_t(f)$ converges. In either case the limit variance is given by*

$$\sigma^2(f) = \lim_{n\to\infty} 2n\langle g_{1/n} - T_{1/n}g_{1/n}, g_{1/n}\rangle. \tag{5}$$

A similar result also holds for discrete-time Markov chains. For $\varepsilon > 0$ set $g_\varepsilon = ((1+\varepsilon)I - Q)^{-1}f$, where I denotes the identity, so that $(1+\varepsilon)g_\varepsilon - Qg_\varepsilon = f$. Then we obtain a decomposition

$$S_n(f) = M_{n,\varepsilon} + \varepsilon S_n(g_\varepsilon) + A_{n,\varepsilon}, \tag{6}$$

where

$$M_{n,\varepsilon} = \sum_{k=1}^{n} (g_\varepsilon(X_k) - (Qg_\varepsilon)(X_{k-1})), \quad A_{n,\varepsilon} = (Qg_\varepsilon)(X_0) - (Qg_\varepsilon)(X_n).$$

Again (6) is called the resolvent representation, and its convergence is defined as in the continuous-time case (just replace t by n in Definition 2.1). The theorem now goes

Theorem 2.3. *Let $(X_n)_{n\geqslant 0}$ be a stationary ergodic Markov chain, defined on a probability space $(\Omega, \mathscr{A}, P)$, with state space $(X, \mathscr{B})$, transition operator Q and stationary initial distribution μ. Let $f \in L_2^0(\mu)$ and $S_n(f) = \sum_{k=1}^n f(X_k)$. Then there exists a martingale approximation to $S_n(f)$ if and only if the resolvent representation of $S_n(f)$ converges. In either case the limit variance is given by*

$$\sigma^2(f) = 2\lim_{\varepsilon\to 0}\langle g_\varepsilon, f\rangle - \|f\|^2. \tag{7}$$

Remark 2.4. Wu and Woodroofe [21] studied approximations by triangular arrays $(M_{n,k})_{k\geqslant 1}$ of martingales (with respect to $\mathscr{F}_k$). If

$$\max_{k\leqslant n} E(S_k - M_{n,k})^2 = \mathrm{o}(\sigma_n^2),$$

where $\sigma_n^2 = ES_n^2(f) \to \infty$, $(M_{n,k})_{k\geqslant 1}$ is called a *martingale approximation scheme*. It is called *stationary* if for each n, $(M_{n,k})_{k\geqslant 1}$ has stationary differences, and *non-triangular* if $M_{n,k} = M_k$ does not depend on n. In this terminology, the martingale approximations of Definition 1.1 are stationary and non-triangular martingale approximation schemes. Wu and Woodroofe [21, Theorem 1], obtained necessary and sufficient conditions for the existence of such martingale approximation schemes. However, the martingale approximation schemes they construct are either stationary or non-triangular but the proof of Theorem 1 in [21] does not yield the existence of martingale approximation schemes which are both stationary and non-triangular. Moreover, the martingale approximation schemes constructed do not imply the CLT. In their paper they also showed that the validity of a conditional version of the CLT is equivalent to a Lindeberg-type condition for the martingale approximation scheme.

Let us turn to the proof of Theorem 2.2. We will first prove two lemmas. Let

$$V_t f = \int_0^t T_s f \, \mathrm{d}s, \quad f \in L_2^0(\mu).$$

Lemma 2.5. *Suppose that* $\|V_n f\| = \mathrm{o}(\sqrt{n})$. Then $\sqrt{\varepsilon}\|g_\varepsilon\| \to 0, \quad \varepsilon \to 0.$

Proof. Observe that $\|V_t f\| \leqslant \|V_{\lfloor t \rfloor}\| + \|f\|$, where $\lfloor t \rfloor$ denotes the integer part of t. Hence there is a non-increasing sequence $\phi_n \to 0$ such that $\|V_t f\|/\sqrt{t} \leqslant \phi_{\lfloor t \rfloor}$, $t \geqslant 1$. Therefore it is easy to find a bounded, continuously differentiable function ψ on $[0, \infty)$ such that $\psi(t) \geqslant \phi_{\lfloor t \rfloor}$, $t \geqslant 1$, and $\psi(t) \to 0$, $t \to \infty$. Using the formula

$$g_\varepsilon = \int_0^\infty \varepsilon \mathrm{e}^{-\varepsilon t} V_t f \, \mathrm{d}t \tag{8}$$

for the resolvent we estimate

$$\begin{aligned} \sqrt{\varepsilon}\|g_\varepsilon\| &\leqslant \int_0^1 \varepsilon^{3/2} \mathrm{e}^{-\varepsilon t} \|V_t f\| \, \mathrm{d}t + \int_1^\infty \varepsilon^{3/2} \mathrm{e}^{-\varepsilon t} \|V_t f\| \, \mathrm{d}t \\ &\leqslant \varepsilon^{3/2} \|f\| + \int_0^\infty \varepsilon^{3/2} \mathrm{e}^{-\varepsilon t} \sqrt{t} \, \psi(t) \, \mathrm{d}t. \end{aligned}$$

Substituting $u = \varepsilon t$ in the second term, we obtain $\int_0^\infty \mathrm{e}^{-u} \sqrt{u} \, \psi(u/\varepsilon) \, \mathrm{d}u$, which tends to 0 as $\varepsilon \to 0$ by dominated convergence. □

Lemma 2.6. *For* $\varepsilon, \delta > 0$ *we have*

$$|\langle g_\varepsilon - g_\delta - T_t(g_\varepsilon - g_\delta), \ g_\varepsilon - g_\delta \rangle| \leqslant 2t(\varepsilon + \delta) \left(\|g_\varepsilon\|^2 + \|g_\delta\|^2 \right). \tag{9}$$

Proof. From Dynkin's formula (3)

$$g_\varepsilon - g_\delta - T_t(g_\varepsilon - g_\delta) = \int_0^t (T_s L g_\delta - T_s L g_\varepsilon) \, \mathrm{d}s.$$

Hence

$$\langle g_\varepsilon - g_\delta - T_t(g_\varepsilon - g_\delta), \ g_\varepsilon - g_\delta \rangle = \int_0^t \langle T_s L g_\delta - T_s L g_\varepsilon, g_\varepsilon - g_\delta \rangle \, \mathrm{d}s. \tag{10}$$

Now since $\delta g_\delta - L g_\delta = f$

$$T_s L g_\delta = \delta T_s g_\delta - T_s f.$$

Therefore

$$\begin{aligned}|\langle T_sLg_\delta - T_sLg_\varepsilon,\, g_\varepsilon - g_\delta\rangle| &= |\langle \delta T_s g_\delta - \varepsilon T_s g_\varepsilon, g_\varepsilon - g_\delta\rangle| \\ &\leqslant \delta\,|\langle T_s g_\delta, g_\delta\rangle| + \varepsilon|\langle T_s g_\varepsilon, g_\varepsilon\rangle| \\ &\quad + \delta\,|\langle T_s g_\delta, g_\varepsilon\rangle| + \varepsilon\,|\langle T_s g_\varepsilon, g_\delta\rangle| \\ &\leqslant \delta\,\|g_\delta\|^2 + \varepsilon\|g_\varepsilon\|^2 + (\varepsilon+\delta)\|g_\varepsilon\|\,\|g_\delta\| \\ &\leqslant 2(\varepsilon+\delta)(\|g_\varepsilon\|^2 + \|g_\delta\|^2).\end{aligned}$$

Applying this inequality in (10) yields the result. □

Proof of Theorem 1. First assume that there exists a martingale approximation $S_t(f) = M_t + A_t$. Since $M_0 = 0$

$$\begin{aligned}\frac{1}{n}\|V_n f\|^2 &= \frac{1}{n}E(E(S_n(f)|\mathscr{F}_0))^2 \\ &= \frac{1}{n}E(E(A_n|\,\mathscr{F}_0))^2 \\ &\leqslant \frac{1}{n}EA_n^2 \to 0, \quad n\to\infty.\end{aligned}$$

Thus $\|V_n f\| = o(\sqrt{n})$, and Lemma 2.5 applies. For any $h \in L_2(\mu)$, from the Schwarz inequality

$$E\left(\int_0^t h(X_s)\,\mathrm{d}s\right)^2 \leqslant E\left(t\int_0^t h(X_s)^2\,\mathrm{d}s\right) = t^2\|h\|^2. \tag{11}$$

From (11) and Lemma 2.5 it follows that $\varepsilon E(S_t(g_\varepsilon))^2 \to 0$, $\varepsilon\to 0$, for any $t>0$. Let us show that $M_{t,\varepsilon}$ converges in $L_2(\Omega,\mathscr{A},P)$ along the sequence $\varepsilon_n = 1/n$ to M_t. Since both $(M_t)_{t\geqslant 0}$ and $(M_{t,\varepsilon})_{t\geqslant 0}$ are martingales with stationary increments with respect to $(\mathscr{F}_t)$, so is $(M_t - M_{t,\varepsilon})$. Therefore

$$\begin{aligned}E(M_{t,\varepsilon} - M_t)^2 &= 1/n\,E(M_{tn,\varepsilon} - M_{tn})^2 \\ &\leqslant 3/n\,EA_{tn}^2 + 3/n\,EA_{tn,\varepsilon}^2 + 3/n\,\varepsilon^2 E(S_{tn}(g_\varepsilon))^2.\end{aligned}$$

By assumption, $1/n\,EA_{tn}^2 \to 0$ as $n\to\infty$. Furthermore, $EA_{tn,1/n}^2 \leqslant 4\|g_{1/n}\|^2$, thus using Lemma 2.5, $1/n\,EA_{tn,1/n}^2 \to 0$. Finally from (11), $E(S_{tn}(g_{1/n}))^2 \leqslant t^2n^2\|g_{1/n}\|^2$, and we obtain the conclusion for the last term. This shows that the resolvent representation converges.

Conversely, assume that the resolvent representation converges. Since $E(S_t(g_\varepsilon)^2) \leqslant t^2\|g_\varepsilon\|^2$, $\varepsilon S_t(g_\varepsilon) \to 0$ as $\varepsilon\to 0$ in $L_2(\Omega,\mathscr{A},P)$. From the resolvent representation (4),

$$A_{t,\varepsilon} = S_t(f) - M_{t,\varepsilon} - \varepsilon S_t(g_\varepsilon).$$

Since for $t>0$ both M_{t,ε_n} and $\varepsilon_n S_t(g_{\varepsilon_n})$ converge as $n\to\infty$ in $L_2(\Omega,\mathscr{A},P)$, and $S_t(f)$ does not depend on n, it follows that A_{t,ε_n} also converges in $L_2(\Omega,\mathscr{A},P)$. Let us show that in fact A_{t,δ_k} converges along an arbitrary sequence $\delta_k\to 0$. Let $n(k)$ be such that

$\varepsilon_{n(k)+1} < \delta_k \leqslant \varepsilon_{n(k)}$. Then $\delta_k \geqslant c\varepsilon_{n(k)}$. From (1) and Lemma 2.6,

$$\begin{aligned} E(A_{t,\delta_k} - A_{t,\varepsilon_{n(k)}})^2 &= 2\langle g_{\delta_k} - g_{\varepsilon_{n(k)}} - T_t(g_{\delta_k} - g_{\varepsilon_{n(k)}}),\ g_{\delta_k} - g_{\varepsilon_{n(k)}}\rangle \\ &\leqslant 4t(\delta_k + \varepsilon_{n(k)})(\|g_{\delta_k}\|^2 + \|g_{\varepsilon_{n(k)}}\|^2) \\ &\leqslant 8t\,\varepsilon_{n(k)}\,\|g_{\varepsilon_{n(k)}}\|^2 + 4t\,(1+1/c)\,\delta_k\,\|g_{\delta_k}\|^2 \to 0, \quad k \to \infty. \end{aligned} \tag{12}$$

Arguing with the resolvent representation as above it follows that both

$$M_{t,\varepsilon} \to M_t \quad \text{and} \quad A_{t,\varepsilon} \to A_t, \quad \varepsilon \to 0 \quad \text{in} \quad L_2(\Omega, \mathscr{A}, P), \quad t \geqslant 0,$$

where $(M_t)_{t\geqslant 0}$ is a martingale with stationary increments with respect to $(\mathscr{F}_t)$, $M_0 = 0$ and $EM_t^2 < \infty$, $EA_t^2 < \infty$ for every t. Thus it remains to show that $EA_t^2/t \to 0$. But

$$EA_t^2 \leqslant 3EA_{t,\varepsilon}^2 + 3E(M_{t,\varepsilon} - M_t)^2 + 3\varepsilon^2 ES_t(g_\varepsilon)^2.$$

Now let $\varepsilon = 1/t$ and proceed as in [17] in the discrete-time situation to obtain the conclusion. Therefore we have a martingale approximation to $S_t(f)$.

Finally let us prove the formula for the limit variance. We have that

$$\sigma^2(f) = EM_1^2 = \lim_{n\to\infty} EM_{1,1/n}^2.$$

Since $(M_{t,1/n})_{t\geqslant 0}$ is a martingale with stationary increments,

$$EM_{1,1/n}^2 = nEM_{1/n,1/n}^2 = nE\left(g_{1/n}(X_{1/n}) - g_{1/n}(X_0) - \int_0^{1/n} (Lg_{1/n})(X_s)\,\mathrm{d}s\right)^2.$$

For any $\varepsilon > 0$, $Lg_\varepsilon = -f + \varepsilon g_\varepsilon$, hence $\|Lg_\varepsilon\| \leqslant \|f\| + \varepsilon\|g_\varepsilon\| \leqslant 2\|f\|$. Thus

$$nE\left(\int_0^{1/n} (Lg_{1/n})(X_s)\,\mathrm{d}s\right)^2 \leqslant 1/n\|Lg_{1/n}\|^2 \overset{n\to\infty}{\to} 0.$$

Since $nE(g_{1/n}(X_{1/n}) - g_{1/n}(X_0))^2 = 2n\langle g_{1/n} - T_{1/n}g_{1/n}, g_{1/n}\rangle$, the formula for $\sigma^2(f)$ follows. The theorem is thus proved. $\square$

Proof of Theorem 2. The proof is similar to that of Theorem 1, therefore we only show how to obtain the formula for the variance.

$$\sigma^2(f) = \lim_{\varepsilon\to 0} EM_{1,\varepsilon}^2 = \lim_{\varepsilon\to 0} (\|g_\varepsilon\|^2 - \|Qg_\varepsilon\|^2).$$

Furthermore,

$$\|g_\varepsilon\|^2 - \|Qg_\varepsilon\|^2 = -2\varepsilon\|g_\varepsilon\|^2 - \varepsilon^2\|g_\varepsilon\|^2 + 2\langle g_\varepsilon, f\rangle - \|f\|^2 + 2\varepsilon\langle g_\varepsilon, f\rangle.$$

All terms vanish as $\varepsilon \to 0$ except for $2\langle g_\varepsilon, f\rangle - \|f\|^2$, and the formula for $\sigma^2(f)$ follows. $\square$

3. Asymptotic normality

In the following theorem we apply Theorem 1 to prove the CLT for stationary Markov processes under a condition which is analogous to that used by Maxwell and Woodroofe [18] in a discrete-time setting.

Theorem 3.1. *Let $(X_t)_{t\geqslant 0}$ be a progressively measurable stationary ergodic Markov process with state space $(X,\mathscr{B})$, strongly continuous contraction semigroup $(T_t)_{t>0}$ and stationary initial distribution μ. Let $f\in L_2^0(\mu)$ and $S_t(f)=\int_0^t f(X_s)\,\mathrm{d}s$. Suppose that f satisfies*

$$\int_1^\infty \|V_t(f)\|/t^{3/2}\,\mathrm{d}t<\infty. \tag{13}$$

Then there exists a martingale approximation to $S_t(f)$. In particular $S_t(f)/\sqrt{t}$ is asymptotically normal with variance $\sigma^2(f)$ given in (5), *and*

$$\sigma^2(f)=\lim_{t\to\infty} ES_t(f)^2/t.$$

Proof. We show that the resolvent approximation converges. Let $\varepsilon_n=1/2^n$. The main point is to show that

$$\sum_{n\geqslant 1}\sqrt{\varepsilon_n}\sup_{\varepsilon_n\leqslant\varepsilon<\varepsilon_{n-1}}\|g_\varepsilon\|<\infty. \tag{14}$$

From this it follows immediately that $\varepsilon\|g_\varepsilon\|^2\to 0,\ \varepsilon\to 0$. Furthermore, from (12)

$$E(A_{t,\varepsilon_{n+1}}-A_{t,\varepsilon_n})^2\leqslant 8t\varepsilon_n\|g_{\varepsilon_n}\|^2+12t\varepsilon_{n+1}\|g_{\varepsilon_{n+1}}\|^2,$$

and since $\sqrt{a+b}\leqslant\sqrt{a}+\sqrt{b},\ a,b\geqslant 0$,

$$\|A_{t,\varepsilon_{n+1}}-A_{t,\varepsilon_n}\|_{L_2(\Omega,\mathscr{A},P)}\leqslant C_1\sqrt{\varepsilon_n}\|g_{\varepsilon_n}\|+C_2\sqrt{\varepsilon_{n+1}}\|g_{\varepsilon_{n+1}}\|.$$

Therefore, from (14)

$$\sum_{n\geqslant 1}\|A_{t,\varepsilon_{n+1}}-A_{t,\varepsilon_n}\|_{L_2(\Omega,\mathscr{A},P)}<\infty,$$

and A_{t,ε_n} converges in $L_2(\Omega,\mathscr{A},P)$ as $n\to\infty$. From the resolvent representation it follows that M_{t,ε_n} also converges. It remains to show (14). Given $\varepsilon>0$ choose n such that $\varepsilon_n\leqslant\varepsilon<\varepsilon_{n-1}$. From (8)

$$\begin{aligned}\|g_\varepsilon\|&\leqslant\varepsilon\int_0^1 \mathrm{e}^{-\varepsilon t}\|V_t(f)\|\,\mathrm{d}t+\varepsilon\int_1^\infty \mathrm{e}^{-\varepsilon t}\|V_t(f)\|\,\mathrm{d}t\\ &\leqslant 2\varepsilon_n\|f\|+2\varepsilon_n\int_1^\infty \mathrm{e}^{-\varepsilon_n t}\|V_t(f)\|\,\mathrm{d}t.\end{aligned}$$

Hence

$$\sum_{n\geqslant 1}\sqrt{\varepsilon_n}\sup_{\varepsilon_n\leqslant\varepsilon<\varepsilon_n}\|g_\varepsilon\|\leqslant 2\sum_{n\geqslant 1}\varepsilon_n^{3/2}\|f\|+2\int_1^\infty\|V_t(f)\|\left(\sum_{n\geqslant 1}\varepsilon_n^{3/2}\mathrm{e}^{-\varepsilon_n t}\right)\mathrm{d}t.$$

But $\sum_{n\geqslant 1}\varepsilon_n^{3/2}e^{-\varepsilon_n t}$ can be seen to be $O(t^{-3/2})$ (cf. [18, p. 715]), and condition (13) implies (14). $\square$

Corollary 3.2. *If* $f \in L_2^0(\mu)$ *satisfies*

$$\int_1^\infty \frac{\|T_t(f)\|}{\sqrt{t}}\,dt<\infty,$$

then it also fulfills (13), *and therefore the conclusions of the theorem remain valid.*

Example 3.3 (*Moving average processes*). We consider the semigroup of translation operators on $L_2[0,\infty)$ given by

$$(T_t f)(u) = f(u+t), \quad f \in L_2[0,\infty)$$

and denote the generator of (T_t) by L. Let $(\xi_t)_{t\in\mathbb{R}}$ be a square-integrable, real-valued process with stationary, independent increments, $E\xi_t = 0$ and $Ed\xi_t^2 = dt$ (cf. [7, p. 111]). Each $f \in L_2[0,\infty)$ gives rise to a stationary, ergodic process $(Y_t)_{t\in\mathbb{R}}$, defined by the stochastic integrals

$$Y_t(f) = Y_t = \int_{-\infty}^t f(t-s)\,d\xi_s.$$

Let $\mathscr{F}_s = \sigma(\xi_u, u\leqslant s)$. We have that

$$E(Y_t|\mathscr{F}_u) = \int_{-\infty}^u f(t-s)\,d\xi_s = \int_{-\infty}^u T_{t-u}f(u-s)\,d\xi_s, \quad t\geqslant u,$$

hence

$$E(E(Y_t|\mathscr{F}_0))^2 = \|T_t f\|^2 = \int_t^\infty f(u)^2\,du.$$

Although $(Y_t)_{t\geqslant 0}$ is not constructed from a Markov process in the way discussed above, these considerations show that our method can still be used with the translation semigroup (T_t) in place of the semigroup of the Markov process. For example, the martingales $(M_{t,\varepsilon})_{t\geqslant 0}$, $\varepsilon>0$, now take the form

$$M_{t,\varepsilon} = Y_t(g_\varepsilon) - Y_0(g_\varepsilon) - \int_0^t Y_s(Lg_\varepsilon)\,ds,$$

where g_ε is formed via the semigroup (T_t). Thus Corollary 3.2 applies, and we obtain that if $f \in L_2[0,\infty)$ satisfies

$$\int_1^\infty \frac{1}{\sqrt{t}}\int_t^\infty f(u)^2\,du\,dt<\infty,$$

then

$$\frac{1}{\sqrt{t}}\int_0^t Y_s\,ds \overset{t\to\infty}{\Rightarrow} N(0,\sigma^2(f)),$$

where $\sigma^2(f)$ is given by (5).

Remark 3.4. Finally suppose that the generator L of the Markov process is a *normal* operator on $L_2^{\mathbb{C}}(\mu)$ ($LL^* = L^*L$). Although conditions for the CLT in the discrete-time case, i.e. in case of a normal transition operator Q, have been studied intensively (cf. Gordin and Lifšic [14], who announced the result and later published the complete proofs in [5, Section IV.7], or Derriennic and Lin [9,10]), there seems to be no continuous-time version in the literature so far. Let us formulate a sufficient condition for the convergence of the resolvent representation and hence for the validity of the CLT for such operators. Given $f \in L_2^0(\mu)$ let ρ_f be the spectral measure of L with respect to f (cf. [3, pp. 123–125]), and let $\sigma(L)$ denote the spectrum of L. One can show that if

$$\int_{\sigma(L)} \frac{1}{|z|} \rho_f(\mathrm{d}z) < \infty, \tag{15}$$

then the resolvent representation converges and hence there exists a martingale approximation to $S_t(f)$, where the limit variance is given by

$$\sigma^2(f) = -2 \int_{\sigma(L)} \frac{1}{z} \rho_f(\mathrm{d}z).$$

A rich class of examples of Markov processes with normal but not necessarily self-adjoint generator arises from convolution semigroups on compact, commutative hypergroups (cf. [4]). Further details can be found in [15]. As pointed out in [10] for the discrete-time case, the condition (15) is weaker than (13) as used in Theorem 3.1.

Acknowledgements

This paper is based on a section of my Ph.D. thesis (Universität Göttingen, 2004), and I would like to thank my supervisor Professor M. Denker for constant encouragement, discussions and advice. Further I thank the referee for pointing out several references.

References

[1] R.N. Bhattacharya, On the functional central limit theorem and the law of the iterated logarithm for Markov processes, Z. Wahrsch. Verw. Gebiete 60 (2) (1982) 185–201.

[2] P. Billingsley, The Lindeberg–Lévy theorem for martingales, Proc. Amer. Math. Soc. 12 (1961) 788–792.

[3] M.Sh. Birman, M.Z. Solomjak, Spectral Theory of Selfadjoint Operators in Hilbert space, D. Reidel Publishing Co., Dordrecht, 1987 (Translated from the 1980 Russian original).

[4] W.R. Bloom, H. Heyer, Harmonic Analysis of Probability Measures on Hypergroups, Walter de Gruyter & Co., Berlin, 1995.

[5] A.N. Borodin, I.A. Ibragimov, Limit theorems for functionals of random walks, Trud. Mat. Inst. Steklov, vol. 195, 1994. (in Russian, English transl. in: Proc. Steklov Inst. Math., 1995).

[6] D.O. Chikin, A functional limit theorem for stationary processes: a martingale approach, Theory Probab. Appl. 34 (4) (1989) 668–678.

[7] H. Cramér, M.R. Leadbetter, Stationary and Related Processes, Wiley, New York, 1967.

[8] A. de Masi, P. Ferrari, S. Goldstein, W. Wick, An invariance principle for reversible Markov processes, J. Statist. Phys. 55 (3–4) (1989) 787–855.
[9] Y. Derriennic, M. Lin, Sur le théorème limite central de Kipnis et Varadhan pour les chaînes réversibles ou normales, C. R. Acad. Sci. Paris Sér. I Math. (323) (1996) 1053–1057.
[10] Y. Derriennic, M. Lin, The central limit theorem for Markov chains with normal transition operators, started at a point, Probab. Theory Related Fields 119 (4) (2001) 508–528.
[11] Y. Derriennic, M. Lin, The central limit theorem for Markov chains started at a point, Probab. Theory Related Fields 125 (1) (2003) 73–76.
[12] M.I. Gordin, H. Holzmann, The central limit theorem for stationary Markov chains under invariant splittings, Stoch. Dyn. 4 (1) (2004) 15–30.
[13] M.I. Gordin, B.A. Lifšic, Central limit theorem for stationary Markov processes, Dokl. Akad. Nauk SSSR 239 (4) (1978) 766–767.
[14] M.I. Gordin, B.A. Lifšic, A remark about a Markov process with normal transition operator, in: Third Vilnius Conference on Probability and Statistics, vol. 1, Vilnius, 1981, pp. 147–148.
[15] H. Holzmann, The central limit theorem for stationary Markov processes with normal generator and applications to random walks on hypergroups, Working Paper, Göttingen University, 2004.
[16] I.A. Ibragimov, A central limit theorem for a class of dependent random variables, Theory Probab. Appl. 8 (1963) 83–89.
[17] C. Kipnis, S.R.S. Varadhan, Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions, Comm. Math. Phys. 104 (1) (1986) 1–19.
[18] M. Maxwell, M. Woodroofe, Central limit theorems for additive functionals of Markov chains, Ann. Probab. 28 (2) (2000) 713–724.
[19] S. Olla, Central limit theorems for tagged particles and for diffusions in random environments, Panor. Syntheses 12 (2001) 75–100.
[20] M. Woodroofe, A central limit theorem for functions of a Markov chain with applications to shifts, Stochastic Process. Appl. 41 (1) (1992) 33–44.
[21] W.B. Wu, M. Woodroofe, Martingale approximations for stationary processes, Ann. Probab. 32 (2) (2004) 1674–1690.