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The central limit theorem for stationary Markov processes with normal generator—with applications to hypergroups

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We extend the central limit theorem for additive functionals of a stationary, ergodic Markov chain with normal transition operator due to Gordin and Lifšic, 1981 [A remark about a Markov process with normal transition operator, In: *Third Vilnius Conference on Probability and Statistics* **1**, pp. 147–48] to continuous-time Markov processes with normal generators. As examples, we discuss random walks on compact commutative hypergroups as well as certain random walks on non-commutative, compact groups.

Keywords: Central limit theorem; Hypergroup; Markov processes; Martingale; Normal generator; Random walk

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1. Introduction

The central limit theorem (CLT) for additive functionals of stationary, ergodic Markov chains has been studied intensively during the last decades. Let $(X_n)_{n\geq 0}$ be a stationary, ergodic Markov chain with state space $(X,\mathcal{B})$, transition operator Q and invariant initial distribution μ. A situation, which is particularly well understood is that in which Q is a normal operator on $L_2^{\mathbb{C}}(\mu)$. This was first considered by Gordin and Lifšic [12]. Denote by L_2^0 the set of real-valued functions with $\int f\,\mathrm{d}\mu = 0$ and let

$$S_n(f) = f(X_1) + \cdots + f(X_n)$$

be the partial sums. Assume that Q is normal and given $f \in L_2^0$ let ρ_f denote the spectral measure of Q with respect to f (cf. [2] for the definition). In [12], it is shown that if $f \in L_2^0$ satisfies

$$\int_{\sigma(Q)} \frac{1}{|1-z|}\,\mathrm{d}\rho_f(z) < \infty, \tag{1}$$

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then $S_n(f)/\sqrt{n}$ is asymptotically normal with variance

$$\sigma^2(f) = \int_{\sigma(Q)} \frac{1-|z|^2}{|1-z|^2}\, \mathrm{d}\rho_f(z). \tag{2}$$

It seems that at that time their result did not receive much attention, and complete proofs were only published later in [4]. Kipnis and Varadhan [16] reproved the result for reversible chains, which correspond to self-adjoint Q, using a different technique and Deriennic and Lin [6] gave a proof for the normal case without use of the spectral theorem. They used the condition $f \in \mathrm{Im}\left(\sqrt{I-Q}\right)$, which is equivalent to equation (1) (cf. [8]). In this paper, we mainly consider continuous-time Markov processes. Let $(X_t)_{t\geq 0}$ be a stationary ergodic Markov process, defined on a probability space $(\Omega, \mathcal{A}, P)$, with state space $(X, \mathcal{B})$, transition probability function $p(t,x,\mathrm{d}y)$ and stationary distribution μ. We assume that the contraction semigroup

$$T_t f(x) = \int_X f(y)p(t,x,\mathrm{d}y), \quad f \in L_2(\mu),$$

is strongly continuous (on $L_2(\mu)$). Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration in $(\Omega, \mathcal{A}, P)$ such that $(X_t)_{t\geq 0}$ is progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$ and satisfies the Markov property

$$E(f(X_t)|\mathcal{F}_u) = T_{t-u}f(X_u), \quad f \in L_2(\mu), \quad 0 \leq u < t.$$

Let L be the generator of $(T_t)_{t\geq 0}$ and $\mathcal{D}(L)$ its domain of definition on $L_2^{\mathbb{C}}(\mu)$. Given $f \in L_2^0, \quad t > 0$ let

$$S_t(f) = \int_0^t f(X_s)\, \mathrm{d}s.$$

Without further assumptions on the generator L, Bhattacharya [1] proved asymptotic normality for $S_t(f)/\sqrt{t}$ (in fact even the functional CLT) if $f \in \mathrm{Im}(L)$. For a reversible Markov process (which corresponds to self-adjoint L), in [16], the CLT under the assumption that $f \in \mathrm{Im}\left(\sqrt{-L}\right)$ is proved.

In this paper, we study the case in which L is a normal operator, i.e. $LL^* = L^*L$. Recall that the generator L is normal if and only if each operator T_t, $t > 0$, of the corresponding semigroup is normal (cf. [19], p. 360). In Section 2, the CLT for Markov processes with normal generator L under a spectral assumption similar to equation (1) is proved, following the method used in [16] for the self-adjoint case. We point out that the method of Gordin and Lifšic [12] for discrete-time chains seems not to be applicable in continuous time. An interesting situation in which the generator L turns out to be normal but not necessarily self-adjoint is that of a convolution semigroup on a compact, commutative hypergroup. In Section 3, we prove a CLT for the corresponding random walks. Random walks on non-commutative compact groups, where the corresponding convolution semigroup is contained in the center of measure algebra, are discussed in Section 4.

2. The central limit theorem

In this section, we will prove the CLT for stationary, ergodic Markov processes with normal generator. Assume that, L is a normal operator on $L_2^{\mathbb{C}}(\mu)$ with spectrum $\sigma(L)$ and for $f \in L_2^0$

denote by $\rho_f(\mathrm{d}z)$ the spectral measure of L with respect to f. Recall that we have $\Re(z) \leq 0$ for each $z \in \sigma(L)$. Consider the condition

$$\int_{\sigma(L)} \frac{1}{|z|} \rho_f(\mathrm{d}z) < \infty. \tag{3}$$

Given $\epsilon > 0$, let $g_\epsilon = (\epsilon I - L)^{-1} f$ be the image under the resolvent mapping. Recall that $g_\epsilon \in \mathcal{D}(L)$, the domain of definition of L, for any $\epsilon > 0$. The norm in $L_2^{\mathbb{C}}(\mu)$ is denoted by $\|\cdot\|$ and the scalar product by $<\cdot,\cdot>$.

Lemma 1. Assume that L is normal and that $f \in L_2^0$ satisfies equation (3). Then

$$\lim_{\epsilon \to 0} \epsilon \langle g_\epsilon, g_\epsilon \rangle = 0 \tag{4}$$

and

$$\lim_{\delta,\epsilon \to 0} \langle g_\epsilon - g_\delta - T_t(g_\epsilon - g_\delta), g_\epsilon - g_\delta \rangle = 0. \tag{5}$$

Proof. In order to show equation (4), from the spectral theorem it follows that

$$\epsilon \langle g_\epsilon, g_\epsilon \rangle = \int_{\sigma(L)} \frac{\epsilon}{|\epsilon - z|^2} \rho_f(\mathrm{d}z).$$

Since, $\Re(z) \leq 0$ for $z \in \sigma(L)$ we estimate

$$|\epsilon - z|^2 = \epsilon^2 + |z|^2 - 2\epsilon \cdot \Re(z) \geq \epsilon^2 + |z|^2 \geq 2\epsilon |z|.$$

Thus equation (4) follows from equation (3) and the dominated convergence theorem. As for equation (5), we have from the spectral theorem

$$\begin{aligned}\langle g_\epsilon - g_\delta - T_t(g_\epsilon - g_\delta), g_\epsilon - g_\delta \rangle &= \int_{\sigma(L)} (1 - e^{zt}) \left[\frac{1}{\epsilon - z} - \frac{1}{\delta - z}\right] \left[\frac{1}{\epsilon - \bar{z}} - \frac{1}{\epsilon - z}\right] \rho_f(\mathrm{d}z) \\ &\leq \int_{\sigma(L)} |1 - e^{zt}| \frac{(\epsilon - \delta)^2}{|\epsilon - z|^2 \cdot |\delta - z|^2} \rho_f(\mathrm{d}z).\end{aligned}$$

We can assume $\epsilon > \delta > 0$. Now $|\epsilon - z|^2 |\delta - z|^2 \geq |z|^2 \epsilon^2$. On $\sigma(L) \cap \{|z| \leq 1\}$ we have $|1 - e^{zt}| \leq |zt| e^t$, and the integrand is dominated by $te^t/|z|$. On $\sigma(L) \cap \{|z| > 1\}$ we have

$$|1 - e^{zt}| \leq 1 + |e^{zt}| = 1 + e^{\Re zt} \leq 2,$$

and the integrand is dominated by $2/|z|^2 \leq 2/|z|$. Again equation (5) follows from equation (3) and the dominated convergence theorem. □

Theorem 1. Let $(X_t)_{t \geq 0}$ be a progressively measurable stationary ergodic Markov process with state space $(X, \mathcal{B})$, strongly continuous contraction semigroup $(T_t)_{t \geq 0}$ and stationary distribution μ. Assume that the generator L is normal on $L_2^{\mathbb{C}}(\mu)$ and that $f \in L_2^0$ satisfies equation (3). Then

$$\frac{S_t(f)}{\sqrt{t}} \overset{t\to\infty}{\Longrightarrow} N(0, \sigma^2(f)),$$

where the limit variance satisfies

$$\sigma^2(f) = \lim_{t\to\infty} E(S_t(f))^2/t = -2\int_{\sigma(L)} \frac{1}{z}\rho_f(\mathrm{d}z).$$

Here, $N(0,\sigma^2)$ denotes the normal law with mean 0 and variance σ^2, and $\Rightarrow$ denotes weak convergence of distributions.

Proof. Consider the decomposition

$$S_t(f) = M_{t,\epsilon} + \epsilon S_t(g_\epsilon) + A_{t,\epsilon},$$

where

$$M_{t,\epsilon} = g_\epsilon(X_t) - g_\epsilon(X_0) - \int_0^t (Lg_\epsilon)(X_s)\,\mathrm{d}s,$$
$$A_{t,\epsilon} = -g_\epsilon(X_t) - g_\epsilon(X_0),$$

and $(M_{t,\epsilon})_{t\geq 0}$ is a martingale with stationary increments and $M_{0,\epsilon} = 0$ for any $\epsilon > 0$. For any $h \in L_2(\mu)$, from the Schwarz inequality,

$$E\left(\int_0^t h(X_s)\,\mathrm{d}s\right)^2 \leq E\left(t\int_0^t h(X_s)^2\,\mathrm{d}s\right) = t^2\|h\|^2. \tag{6}$$

From equations (4) and (6), it follows that $\epsilon^2 ES_t(g_\epsilon)^2 \to 0$. Furthermore, since

$$E(A_{t,\epsilon} - A_{t,\delta})^2 = 2 < g_\epsilon - g_\delta - T_t(g_\epsilon - g_\delta), g_\epsilon - g_\delta >,$$

the convergence of $A_{t,\epsilon}$ to some A_t as $\epsilon \to 0$ follows directly from equation (5) via the Cauchy criterion. Since, $M_{t,\epsilon} = \epsilon S_t(g_\epsilon) + A_{t,\epsilon} - S_t(f)$, $M_{t,\epsilon}$ also converges to a limit M_t, which is also a martingale with stationary increments, and $S_t(f) = M_t + A_t$. Using equation (4), it is easy to show (see Kipnis and Varadhan [16]) that $EA_t^2/t \to 0$ as $t \to \infty$. Asymptotic normality follows from the CLT for martingales with stationary increments. This result is well-known for discrete-time martinagles; see Chikin [5] for a careful discussion of the continuous-time case. Finally, let us prove the formula for $\sigma^2(f)$. From [14],

$$EM_1^2 = \sigma^2(f) = \lim_{n\to\infty} 2n < g_{1/n} - T_{1/n}g_{1/n}, g_{1/n} > .$$

Now

$$2n < g_{1/n} - T_{1/n}g_{1/n}, g_{1/n} >= 2\int_{\sigma(L)} \frac{1 - e^{z/n}}{1/n}\frac{1}{|1/n - z|^2}\,\mathrm{d}\rho_f(z).$$

The intergrand converges to $-1/\bar{z}$, and by an application of the dominated convergence theorem, which can be justified as above the formula for $\sigma^2(f)$ follows. This finishes the proof of the theorem. □

Remark 1. A functional CLT for Markov chains with normal transition operator, started at a point, was proved in [7] under a spectral assumption slightly stronger than (3). It would be of some interest to obtain a similar result for continuous-time Markov processes.

Remark 2. Kipnis and Varadhan [16] in fact obtained the functional central limit theorem for reversible Markov processes under the condition (3). A simpler proof of the functional part was given by Olla [18]. It is possible to deduce from his results that if

$$-\int_{\sigma(L)} \frac{1}{\Re z}\, \mathrm{d}\rho_f(z) < \infty,$$

then the functional CLT holds in case of a stationary Markov process with normal generator. Whether such a result is already true under the milder spectral assumption (3) remains an open problem.

3. Random walks on compact commutative hypergroups

In this section, we apply Theorem 1 to random walks on compact commutative hypergroups. Roughly speaking, a hypergroup is a Hausdorff space H such that the space of regular finite Borel measures $\mathcal{M}_{\mathrm{b}}(H)$ can be equipped with a convolution operation, which preserves the probability measures. Axiomatic schemes for this concept were first introduced by Dunkl [9] and Jewett [15]. Since then hypergroups have been investigated intensively, due to the rich variety of examples, and a rather general notion of hypergroups has become standard in the literature. Let H be a locally compact Hausdorff space. We denote by $\mathcal{M}_{\mathrm{b}}(H)$ the space of regular finite Borel measures and by $\mathcal{M}_1(H)$ the subset of regular probabilities. Our definition of a hypergroup is taken from Bloom and Heyer [3].

Definition 1. H is called a hypergroup if the space $(\mathcal{M}_{\mathrm{b}}(H), +)$ admits a second binary operation $*$ such that the following conditions are satisfied.

1. $(\mathcal{M}_{\mathrm{b}}(H), +, *)$ is an algebra.
2. For any $x, y \in H$, $\delta_x * \delta_y \in \mathcal{M}_1(H)$ and supp $(\delta_x * \delta_y)$ is compact (here, δ_x denotes the Dirac measure at $x \in H$).
3. The mappings $(x,y) \mapsto \delta_x * \delta_y$ and $(x,y) \mapsto \mathrm{supp}(\delta_x * \delta_y)$ of $H \times H$ are continuous with respect to the weak topology and the Michael topology, respectively.
4. There exists an involution $x \mapsto \bar{x}$ of H such that $\overline{\delta_x * \delta_y} = \delta_{\bar{y}} * \delta_{\bar{x}}$ for all $x,y \in H$, where $\bar{\nu}$ denotes the image of $\nu \in \mathcal{M}_{\mathrm{b}}(H)$ under the involution.
5. There exists an element $e \in H$ such that $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all $x \in H$, and such that $e \in \mathrm{supp}(\delta_x * \delta_y)$ if and only if $y = \bar{x}$, $x,y \in H$.

The hypergroup H is called *commutative* if $(\mathcal{M}_{\mathrm{b}}(H), +, *)$ is a commutative algebra. In the following, let H be a commutative hypergroup. The *x-translate* of a function $f \in C_{\mathrm{c}}(H)$ is defined by

$$\tau_x f(y) = f(x*y) = \int_H f\, \mathrm{d}(\delta_x * \delta_y).$$

A measure $\nu \in \mathcal{M}_b(H)$ is called *invariant* if

$$\int_H \tau_x f \,\mathrm{d}\mu = \int_H f \,\mathrm{d}\mu, \quad f \in C_c(H), \quad x \in H.$$

A *compact* hypergroup (i.e. H is a compact) always admits a unique invariant measure $\mu \in \mathcal{M}_1(H)$ (cf. [3], p. 40), and we have the formula (cf. [3], p. 34)

$$\int_H f(x*y)g(y)\,\mathrm{d}\mu(y) = \int_H f(y)g(\bar{x}*y)\,\mathrm{d}\mu(y) \quad \forall f, g \in L_2^{\mathbb{C}}(\mu). \tag{7}$$

Furthermore, translation can be extended to the space $L_2^{\mathbb{C}}(\mu)$. The *convolution* of a function $f \in L_2^{\mathbb{C}}(\mu)$ and a measure $\nu \in \mathcal{M}_b(H)$ is defined by

$$f*\nu(x) = \int_K f(x*\bar{y})\,\mathrm{d}\nu(y).$$

A non-zero, continuous function $\chi : H \to \mathbb{C}$ is called a *character* if

$$\chi(x*\bar{y}) = \chi(x)\overline{\chi(y)}, \quad x, y \in H.$$

It follows that $\chi(e) = 1$, $|\chi(x)| \leq 1$ and $\chi(\bar{x}) = \overline{\chi(x)}$. The set of characters is denoted by $\hat{H}$. If H is compact and commutative, $\hat{H}$ is discrete (with respect to the topology of uniform convergence), and forms an orthogonal basis of $L_2^{\mathbb{C}}(\mu)$ (cf. [9], p. 340). The *Fourier transform* of a function $f \in L_2^{\mathbb{C}}(\mu)$ and of a measure $\nu \in \mathcal{M}_b(H)$ are defined respectively by

$$\hat{f}, \hat{\nu} : \hat{H} \to \mathbb{C}, \quad \hat{f}(\chi) = \int_H f\overline{\chi}\,\mathrm{d}\mu, \quad \hat{\nu}(\chi) = \int_H \overline{\chi}\,\mathrm{d}\nu.,$$

The *Plancherel measure* on $\hat{H}$ is given by $\pi = \sum_{\chi \in H} c(\chi)\delta_\chi$, where δ_χ is the Dirac measure at x and

$$c(\chi) = \left(\int_H |\chi|^2 \,\mathrm{d}\mu\right)^{-1}.$$

Furthermore, we have the Plancherel formula and the inversion formula (cf. [3], pp. 86, 91).

Firstly, let us consider discrete-time random walks. Let $Q \in \mathcal{M}_1(H)$ be a probability measure on H. Then, we can define a Markov kernel Q on $L_2^{\mathbb{C}}(\mu)$ by letting $Qf(x) = f*Q(x)$. Using the translation invariance of the Haar measure one shows that this Markov kernel preserves μ. Now we are in the position to state the following result.

Theorem 2. Let H be a compact, commutative hypergroup with Haar measure μ. Let $Q \in \mathcal{M}_1(H)$ and let $(X_n)_{n\geq 0}$ be a random walk in H with transition operator Q and stationary distribution μ. Suppose that 1 is a simple eigenvalue of Q and that $f \in L_2^0$ satisfies

$$\sum_{\chi \in \hat{H}} \frac{1}{|1 - \hat{Q}(\chi)|} c(\chi)|\hat{f}(\chi)|^2 < \infty.$$

Then $S_n(f)/\sqrt{n}$ is asymptotically normally distributed, and the limit variance is given by

$$\sigma^2(f) = \sum_{\chi \in \hat{H}} \frac{1 - |\hat{Q}(\chi)|^2}{|1 - \hat{Q}(\chi)|^2} c(\chi)|\hat{f}(\chi)|^2.$$

Proof. We want to apply condition (1), as obtained by Gordin and Lifšic [12]. It is well-known that the chain $(X_n)_{n\geq 0}$ is ergodic if and only if 1 is a eigenvalue of Q. Now let us show that Q is a normal operator. To this end, using equation (7) the following is easily shown.

$$\int_H (Qf)(x)\overline{g(x)}\,\mathrm{d}\mu(x) = \int_H f(x) \int_H \overline{g(x*y)}\,\mathrm{d}Q(y)\,\mathrm{d}\mu(x), \quad f, g \in L_2^{\mathbb{C}}(\mu).$$

Therefore, the adjoint operator is given by $(Q^* g)(x) = \int g(x * y)\,\mathrm{d}Q(y)$, i.e. by convolution with respect to the measure $\bar{Q}$. By commutativity it follows that Q is normal. Furthermore, we have that

$$\chi * Q = \hat{Q}(\chi)\chi, \quad \chi \in \hat{H}. \tag{8}$$

Therefore, Q has a discrete spectrum and each χ is an eigenvector with eigenvalue $\hat{Q}(\chi)$. The theorem now follows from equations (1) and (2). □

Remark 3. Related results on the central limit theorem for random walks on hypergroups, where H is a non-compact interval or the lattice $\mathbb{Z}$ or $\mathbb{Z}_+$, can be found in [11].

Now let us consider continuous-time random walks. A *convolution semigroup* $(Q_t)_{t>0} \subset \mathcal{M}_1(H)$ is a family of probability measures such that $Q_t * Q_s = Q_{s+t}$. It is called *e-continuous* (or simply *continuous*) if $\lim_{t\to 0} Q_t = \delta_e$ in the topology of weak convergence. For every e-continuous convolution semigroup there exists a negative definite function $\psi \in N_B^{(s)}(\hat{H})$ (see [3], p. 334), called the *exponent* of the convolution semigroup, such that $\hat{Q}_t = \exp(-t\psi)$. Given an e-continuous convolution semigroup, we obtain a contraction semigroup by letting $T_t = f * Q_t$ $f \in L_2^{\mathbb{C}}(\mu)$ (cf. [3], p. 427). This semigroup commutes with translations, and gives rise to a stationary Markov process $(X_t)_{t\geq 0}$ with stationary distribution μ. We have the following

THEOREM 3. Let H be a compact, commutative hypergroup with Haar measure μ. Let $(Q_t)_{t>0}$ be an e-continuous convolution semigroup with exponent $\psi \in N_B^{(s)}(\hat{H})$ and let $(X_t)_{t\geq 0}$ be the corresponding continuous-time random walk with semigroup T_t, generator L, and stationary distribution μ. Suppose that 0 is a simple eigenvalue of L and that $f \in L_2^0$ satisfies

$$\sum_{\chi\in\hat{H}} \frac{1}{|\psi(\chi)|} c(\chi)|\hat{f}(\chi)|^2 < \infty. \tag{9}$$

Then $S_t(f)/\sqrt{t}$ is asymptotically normally distributed with limit variance

$$\sigma^2(f) = 2\sum_{\chi\in\hat{H}} \frac{1}{\psi(\chi)} c(\chi)|\hat{f}(\chi)|^2.$$

Proof. First let us show that the semigroup (T_t) is strongly continuous. In fact, the Fourier transform gives rise to the contraction semigroup on $L_2^{\mathbb{C}}(\hat{H}, \pi)$ given by the multiplication operators $M_t F = \exp(-t\psi)F$, $F \in L_2^{\mathbb{C}}(\hat{H}, \pi)$. Such contraction semigroups are always strongly continuous (cf. Nagel and Schlotterbeck [17], p. 8), and their generator is the densely-defined multiplication operator $\hat{L}F = -\psi F$. Thus from the Fourier isometry, it follows that the generator L of (T_t) is also densely defined with domain

$$\mathcal{D}(L) = \left\{ f \in L_2^{\mathbb{C}}(\hat{H}, \mu) : \psi\hat{f} \in L_2^{\mathbb{C}}(\hat{H}, \pi) \right\},$$

and

$$\widehat{(Lf)} = -\psi\hat{f}, \quad f \in \mathcal{D}(L).$$

For $f = \chi$ with $\chi \in \hat{H}$ this gives

$$\widehat{(L\chi)}(\gamma) = -\psi(\chi)c(\chi)^{-1}1_{\{\chi\}}(\gamma), \quad \chi, \gamma \in \hat{H}.$$

From the inversion theorem ([3], pp. 89–92) we get that

$$L\chi = -\psi(\chi)\chi.$$

The theorem follows from theorem 1. □

Remark 4. Observe that L is self-adjoint if and only if $Q_t = \bar{Q}_t$ for all $t > 0$.

Example 1. (*Compact Abelian groups*). In this examples we illustrate the use of Theorems 2 and 3 by considering random walks on a separable compact Abelian group G. Let Γ denote the dual group of G and let μ_G be the normalized Haar measure. It is well known that characters form an orthonormal basis of $L_2^{\mathbb{C}}(G)$. There is a hypergroup structure on G given by the usual convolution, i.e. $\delta_x + \delta_y = \delta_{x+y}$. Thus Haar measure on the hypergroup is the usual Haar measure on G, and the characters of the hypergroup are given by the characters of the group. Theorems 2 and 3 apply, and $c(\chi) = 1$ for all $\chi \in \Gamma$. In discrete time, this example was studied by Gordin & Lifšic ([4], pp. 171, 72). Given an e-continuous convolution semigroup, the generating functional ψ can be decomposed as follows:

$$\psi = \psi_1 + \psi_2 + \psi_3,$$

where ψ_1 is a continuous primitive form, ψ_2 a continuous square form, and ψ_3 is given in terms of the Lévy function and the Lévy measure (see Heyer [13], pp. 70, 308). Let us consider the one-dimensional torus $\mathcal{T}^1$, where characters are of the form $\chi_n(\theta) = e^{in\theta}$, $\theta \in [0, 2\pi)$. In this case (cf. Zimple [20], p. 493),

$$\psi_1(\chi_n) = -ian, \quad \psi_2(\chi_n) = bn^2, \quad a \in \mathbb{R}, \quad b \geq 0.$$

If $\psi = \psi_1$, $X_t = e^{iat}$ is a deterministic motion. As can be expected, (9) is satisfied for any $f \in L_2^0$ but $\sigma^2(f) = 0$. If $\psi = \psi_2$, the Q_t are wrapped Gaussian distributions with densities

$$q_t(\theta) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{-tn^2 b}\cos(n\theta).$$

Equation (9) is also satisfied for any $f \in L_2^0$, and $\sigma^2(f) \neq 0$ if $f \neq 0$ (and $b \neq 0$). Notice that L is self-adjoint in this case. If the Lévy measure α is bounded, then

$$\psi_3(\chi_n) = \int_{G\setminus\{e\}} (1 - \chi_n(\theta))\,\mathrm{d}\alpha(\theta).$$

In this case (as well as in the case of general ψ), asymptotic normality depends on the Fourier expansion of $f \in L_2^0$.

4. Random walks on compact, non-Abelian groups

In this section we show how to apply Theorem 1 to certain random walks on compact, possibly non-Abelian groups.

Let G be a compact, separable group with normalized Haar measure μ_G and let $\hat{G}$ denote the set of equivalence classes of irreducible unitary representations of G. If $\alpha \in \hat{G}$, we let α also stand for some representative of this equivalence class, acting on a space V_α of finite dimension n_α. We have the orthogonal Hilbert space decomposition

$$L_2^{\mathbb{C}}(G) = \oplus_{\alpha \in \hat{G}} H_\alpha, \quad H_\alpha = \{g \mapsto \mathrm{tr}(\alpha(g)C), \quad g \in G, \quad C \in \mathrm{End}(V_\alpha)\}, \quad \alpha \in \hat{G},$$

where $\mathrm{tr}(C)$ denotes the trace of the endomorphism C (cf. Ref. [10]). The orthogonal projection of $f \in L_2^{\mathbb{C}}(G)$ to H_α is given by $n_\alpha f_\alpha$, where $f_\alpha = f * \chi_\alpha = \chi_\alpha * f$, and χ_α is the character of α.

Let H be the set of conjugacy classes with the quotient topology. There is a one-to-one correspondence between $\mathcal{M}_b(H)$ and $\mathcal{Z}(\mathcal{M}_b(G))$, the center of $\mathcal{M}_b(G)$. Therefore H can be equipped with a commutative hypergroup structure, and Theorems 2 and 3 apply to random walks on H. Explicitly, the characters of H are given by the normalized characters of the group $\gamma_\pi = \chi_\pi / n_\alpha$.

We want to extend this result to functions which are not necessarily conjugation-invariant. Notice that $Q \in \mathcal{Z}(\mathcal{M}_b(G))$ is ergodic on $L_2^{\mathbb{C}}(G)$ if and only if it is ergodic on $L_2^{\mathbb{C}}(H, \mu)$, since 0 is either a simple or multiple eigenvalue in both cases.

Theorem 4. Let G be a compact, separable, non-Abelian group and let Q be a probability on G. Suppose that $Q \in \mathcal{Z}(\mathcal{M}^b(G))$ and that Q, as a convolution operator, has 1 as a simple eigenvalue. Let $(X_n)_{n \geq 0}$ be a random walk on G with transition operator Q and stationary distribution μ_G. If $f \in L_2^0$ satisfies

$$\sum_{\alpha \in \hat{G}} \frac{1}{|1 - \hat{Q}(\chi_\alpha)/n_\alpha|} n_\alpha^2 \|f_\alpha\|^2 < \infty,$$

then $S_n(f)/\sqrt{n}$ is asymptotically normally distributed, where the limit variance is given by

$$\sum_{\alpha \in \hat{G}} \frac{1 - |\hat{Q}(\chi_\alpha)/n_\alpha|^2}{|1 - \hat{Q}(\chi_\alpha)/n_\alpha|^2} n_\alpha^2 \|f_\alpha\|^2 < \infty.$$

Proof. Since $Q \in \mathcal{Z}(\mathcal{M}_b(G))$, from (8) we obtain $Q * \gamma_\alpha = \hat{Q}(\gamma_\alpha)\gamma_\alpha$ or $Q * \chi_\alpha = \hat{Q}(\chi_\alpha)/n_\alpha \chi_\alpha$. Given any $f \in L_2^{\mathbb{C}}(G)$ and $\alpha \in \hat{G}$, we have since $Q \in \mathcal{Z}(\mathcal{M}^b(G))$,

$$Q * f_\alpha = Q * f * \chi_\alpha = f * Q * \chi_\alpha = \hat{Q}(\chi_\alpha)/n_\alpha f * \chi_\alpha = \hat{Q}(\chi_\alpha)/n_\alpha f_\alpha.$$

Therefore, each space H_α is an eigenspace of Q with eigenvalue $\hat{Q}(\chi_\alpha)/n_\alpha$ and in particular, Q is a normal operator. The theorem follows from condition (1), due to Gordin and Lifšic [12]. □

A similar result can be formulated for e-continuous convolution semigroups in $\mathcal{Z}(\mathcal{M}^b(G))$.

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References

[1] Bhattacharya, R.N., 1982, On the functional central limit theorem and the law of the iterated logarithm for Markov processes. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, **60**, 185–201.
[2] Birman, M.Sh. and Solomjak, M.Z., 1987, *Spectral Theory of Selfadjoint Operators in Hilbert Space* (Dordrecht: D. Reidel Publishing Co.).
[3] Bloom, W.R. and Heyer, H., 1995, *Harmonic Analysis of Probability Measures on Hypergroups* (Berlin: Walter de Gruyter and Co.).
[4] Borodin, A.N. and Ibragimov, I.A., 1994, *Limit Theorems for Functionals of Random Walks* (Proc. Steklov Inst. Math.).
[5] Chikin, D.O., 1989, A functional limit theorem for stationary processes: a martingale approach. *Theory of Probability and its Applications*, **34**, 668–678.
[6] Derriennic, Y. and Lin, M., 1996, Sur le théorème limite central de Kipnis et Varadhan pour les chaînes réversibles ou normales. *Comptes Rendus de'Academie des Sciences. Paris Série I. Mathematique*, **323**, 1053–1057.
[7] Derriennic, Y. and Lin, M., 2001, The central limit theorem for Markov chains with normal transition operators, started at a point. *Probability Theory and Related Fields*, **119**, 508–528.
[8] Derriennic, Y. and Lin, M., 2001, Fractional Poisson equations and ergodic theorems for fractional coboundaries. *Israel Journal of Mathematics*, **123**, 93–130.
[9] Dunkl, C.F., 1973, The measure algebra of a locally compact hypergroup. *Transactions of the American Mathematical Society*, **179**, 331–348.
[10] Dunkl, C.F., Donald, E. and Ramirez, D.E., 1971, *Topics in Harmonic Analysis* (New York: Appleton-Century-rofts).
[11] Gallardo, L., 2002, A multidimensional central limit theorem for random walks on hypergroups. *Stochastics and Stochastics Reports*, **73**, 1–23.
[12] Gordin, M.I. and Lifšic, B.A., 1981, A remark about a Markov process with normal transition operator. In *Third Vilnius Conference on Probability and Statistics*, **1**, pp. 147–148.
[13] Heyer, H., 1977, *Probability Measures on Locally Compact Groups* (Berlin: Springer-Verlag).
[14] Holzmann, H., 2005, Martingale approximations for continuous-time and discrete-time stationary Markov processes. *Stochastic Processes and their Applications,* to appear.
[15] Jewett, R.I., 1975, Spaces with an abstract convolution of measures. *Advances in Mathematics*, **18**, 1–101.
[16] Kipnis, C. and Varadhan, S.R.S., 1986, Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Communications in Mathematical Physics,* **104**, 1–19.
[17] Nagel, R. and Schlotterbeck, U., 1986, *One-Parameter Semigroups of Positive Operators*, Lecture Notes in Mathematics 1184 (Berlin: Springer-Verlag), pp. 1–24.
[18] Olla, S., 2001, Central limit theorems for tagged particles and for diffusions in random environments. *Panoramas et Syntheses*, **12**, 75–100.
[19] Rudin, W., 1973, *Functional Analysis* (New York: McGraw-Hill Book Co.).
[20] Zimpel, Z., 1990, Invariant Markov processes on compact groups and correlation functions. *Physica A*, **164**, 485–502.