Two-component mixtures with independent coordinates as conditional mixtures: Nonparametric identification and estimation

Technical report

Daniel Hohmann **Hajo Holzmann**

Philipps-Universität Marburg

March 6, 2013

1 Identification

Consider the conditional mixture

$$F(y|B) = (1 - \pi(B))F_0(y) + \pi(B)F_1(y), \qquad y \in \mathbb{R},\ B \in \mathcal{B}^p, \tag{1.1}$$

with mixture weight function π and component distribution functions F_0 and F_1. To identify the components in (1.1), assume that

A1. 1. there exist $B_0, B_1 \in \mathcal{B}^p$ such that $0 < \pi(B_0), \pi(B_1) < 1$, $\pi(B_0) \neq \pi(B_1)$.
2. there exists a $y_0 \in \mathbb{R}$ such that $F_1(y_0) \neq F_0(y_0)$.

Further, consider the following tail conditions.

C1. $\lim_{y\to-\infty} F_1(y)/F_0(y) = 0$

C2. $\lim_{y\to+\infty}(1 - F_0(y))/(1 - F_1(y)) = 0$

C3. $\lim_{y\to+\infty} \tilde{F}_0(y)/\tilde{F}_1(y) = 0$

Hohmann and Holzmann (2013, Theorem 3) show that mixture (1.1) is identifiable under A1 and assuming C1 and C2/C3. The following example shows that it does not suffice to impose only one of the tail conditions above.

Example 1. Assume that mixture (1.1) is identifiable in the sense of Theorem 3 in Hohmann and Holzmann (2013). Let $\pi_2 : \mathcal{B}^p \to [0, 1]$ be a different weight function such that $\pi_2(B_0) < \pi_1(B_0)$, and set

$$\pi_2(B) = 1 - \frac{(1 - \pi_1(B))(1 - \pi_2(B_0))}{1 - \pi_1(B_0)}, \qquad B \in \mathcal{B}\backslash\{B_0\}. \tag{1.2}$$

Further, set $G_1 = F_1$, and define G_0 according to

$$G_0(y) = \frac{1-\pi_1(B_0)}{1-\pi_2(B_0)}\, F_0(y) + \Big(1 - \frac{1-\pi_1(B_0)}{1-\pi_2(B_0)}\Big) F_1(y).$$

Then, G_0 is indeed a distribution function due to $\pi_1(B_0) > \pi_2(B_0)$, and by construction the ratio $\rho = (1-\pi_1(B))/(1-\pi_2(B))$ does not depend on B, so for all $y \in \mathbb{R}$ and $B \in \mathcal{B}^p$ we obtain that

$$G(y|B) = (1-\pi_2(B))G_0(y) + \pi_2(B)G_1(y) = F(y|B).$$

Also, G_0 and G_1 meet C1 since

$$\frac{G_1(y)}{G_0(y)} = \frac{F_1(y)/F_0(y)}{\rho + (1-\rho)F_1(y)/F_0(y)} \longrightarrow 0 \qquad \text{as } y \to -\infty,$$

while, in general, they satisfy neither of the conditions C2 and C3. □

The next example shows that also the role of the conditioning events $\{Z \in B_0\}$ and $\{Z \in B_1\}$ is important for the nonparametric identification of a two-component mixture. In fact, even with known mixture proportion π, regularity conditions such as C1 and C2 do not provide the identification of an ordinary mixture

$$F(y) = (1-\pi)F_0(y) + \pi F_1(y). \tag{1.3}$$

Example 2. Assume that F_0 and F_1 in (1.3) are absolutely continuous with densities f_0 and f_1, respectively, and assume that there exist $a, b \in \mathbb{R}$, $a < b$, such that F_0 is strictly concave on the interval (a, b) and

$$f_0(y) + \frac{\pi}{1-\pi}\, f_1(y) \geq \frac{F_0(b) - F_0(a)}{b-a}\,, \qquad a \leq y \leq b. \tag{1.4}$$

Set $G_0 = F_0 \mathbf{1}_{[a,b)^\complement} + F_a^b$ and $G_1 = F_1 + \frac{1-\pi}{\pi}(F_0\mathbf{1}_{[a,b)} - F_a^b)$, where

$$F_a^b(y) = \Big(\frac{(y-a)F_0(b) - (y-b)F_0(a)}{b-a}\Big)\mathbf{1}_{[a,b)}(y).$$

(1.4) guarantees that G_1 is non-decreasing and thus a distribution function. Now G_0 and G_1 adopt C1 and C2 from F_0 and F_1, and the mixture $G(y) = (1-\pi)G_0 + \pi G_1(y)$ satisfies $G = F$. □

2 Estimating quotients in the tails

Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be mutually independent sequences of i.i.d. observations with distribution functions F and G, respectively, and assume that

$$F(y)/G(y) \longrightarrow \theta \qquad \text{as } y \to -\infty \tag{2.1}$$

and

$$\widetilde{F}(y), \widetilde{G}(y) \longrightarrow 0, \quad \widetilde{F}(y)/\widetilde{G}(y) \longrightarrow \eta \qquad \text{as } y \to \infty \tag{2.2}$$

hold for some $\theta > 0$ and $\eta \in \mathbb{C}\backslash\{0\}$, where $\widetilde{F}$ and $\widetilde{G}$ denote the characteristic functions of F and G. We shall construct asymptotically normal estimators of θ and η. In the following, suppose that l_n and m_n are sequences in $\mathbb{N}$ such that $l_n, m_n \asymp n$ as $n \to \infty$.

To estimate η in (2.2), let

$$\eta_n = \widetilde{F}_n(h_n)/\widetilde{G}_n(h_n), \qquad \widetilde{F}_n(y) = \frac{1}{l_n}\sum_{k=1}^{l_n} \exp(iyX_k), \qquad \widetilde{G}_n(y) = \frac{1}{m_n}\sum_{k=1}^{m_n} \exp(iyY_k),$$

with h_n a sequence tending to infinity. Decompose

$$\eta_n - \eta = (\eta_n - \bar{\eta}_n) + (\bar{\eta}_n - \eta), \qquad \bar{\eta}_n = \widetilde{F}(h_n)/\widetilde{G}(h_n).$$

In order to handle the "variance term", write

$$\sqrt{r_n}(\eta_n - \bar{\eta}_n) = \frac{\sqrt{r_n/m_n}}{\widetilde{G}_n(h_n)}\Big(\sqrt{m_n/l_n}\,\widetilde{\mathbf{F}}_n(h_n) - \bar{\eta}_n \widetilde{\mathbf{G}}_n(h_n)\Big), \tag{2.3}$$

where $\widetilde{\mathbf{F}}_n = \sqrt{l_n}(\widetilde{F}_n - \widetilde{F})$ and $\widetilde{\mathbf{G}}_n = \sqrt{m_n}(\widetilde{G}_n - \widetilde{G})$ are the characteristic processes and $r_n \to \infty$. Assume that r_n satisfies

$$r_n/n \to 0, \quad r_n/\sqrt{n} \to \infty \qquad \text{as } n \to \infty, \tag{2.4}$$

and that $h_n \to_p \infty$ is chosen such that

$$|\widetilde{G}_n(h_n)| = \sqrt{r_n/m_n}(1 + o_P(1)). \tag{2.5}$$

We shall use strong approximations of the characteristic processes by

$$\mathbf{C}(y) = \int \exp(iyx)\,\mathbf{B}(F(dx)) \tag{2.6}$$

for $\widetilde{\mathbf{F}}_n(y)$, and similarly for $\widetilde{\mathbf{G}}_n$. In order that these processes are sample-continuous and that strong approximations work, some conditions on F and G are required, see Csörgő (1981). We shall adopt the following sufficient condition: Assume that there exists $\gamma > 0$

such that

$$y^{\gamma} H(-y) + y^{\gamma}(1 - H(y)) = O(1) \quad \text{as } y \to \infty, \quad H = F, G. \tag{2.7}$$

Finally, we assume that there also exists a non-random sequences $t_n \to \infty$ such that

$$t_n = o\big(n^{\gamma/(2\gamma+4)}(\log n)^{-(\gamma+1)/(\gamma+2)}\big), \tag{2.8}$$

$$|h_n - t_n| = o_P(1), \tag{2.9}$$

$$|\widetilde{G}(h_n) - \widetilde{G}(t_n)| = o_P\big(\sqrt{r_n/m_n}\big), \tag{2.10}$$

with γ determined by (2.7).

To estimate θ in (2.1), let

$$\theta_n = F_n(h_n)/G_n(h_n), \qquad F_n(y) = \frac{1}{l_n}\sum_{k=1}^{l_n} \mathbf{1}_{\{X_k \le y\}}, \quad G_n(y) = \frac{1}{m_n}\sum_{k=1}^{m_n} \mathbf{1}_{\{Y_k \le y\}},$$

where the level h_n is specified below. Write

$$\theta_n - \theta = (\theta_n - \bar{\theta}_n) + (\bar{\theta}_n - \theta), \qquad \bar{\theta}_n = F(h_n)/G(h_n).$$

Again assume that $r_n \to \infty$ satisfies (2.4), and that $h_n \to_p -\infty$ is chosen such that

$$G_n(h_n) = r_n/m_n + o_P(r_n/n) = r_n/m_n\,(1 + o_P(1)). \tag{2.11}$$

(2.11) is satisfied if we choose in particular $h_n = Y_{m_n(\lfloor r_n \rfloor)}$, where $\lfloor r_n \rfloor$ is the largest integer smaller than r_n, and where $Y_{m_n(\lfloor r_n \rfloor)}$ denotes the $\lfloor r_n \rfloor$-th largest order statistic of the sample $Y_1, \ldots, Y_{m_n}$, since $G_n(h_n) = \lfloor r_n \rfloor / m_n = r_n/m_n(1 + o(1))$.

Theorem 3. *Suppose that (2.4), (2.5) and (2.7)-(2.11) hold. If there exists $\tau > 0$ such that $m_n/l_n \to \tau$, then*

$$\sqrt{r_n}\begin{pmatrix} \theta_n - \bar{\theta}_n \\ \operatorname{Re}(\eta_n - \bar{\eta}_n) \\ \operatorname{Im}(\eta_n - \bar{\eta}_n) \end{pmatrix} \rightsquigarrow \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 2(\tau\theta + \theta^2) & 0 & 0 \\ 0 & \tau + |\eta|^2 & 0 \\ 0 & 0 & \tau + |\eta|^2 \end{pmatrix}\right).$$

The proof of Theorem 3 proceeds in several steps. The asymptotic normality of $\sqrt{r_n}(\eta_n - \bar{\eta}_n)$ was shown in Hohmann and Holzmann (2013). We continue by showing that

$$\sqrt{r_n}\big(\theta_n - \bar{\theta}_n\big) \rightsquigarrow \mathcal{N}\big(0, \tau\theta + \theta^2\big), \tag{2.12}$$

for which we need the following additional results.

Lemma 4. *Let $r_n \to \infty$, $r_n/n \to 0$ as $n \to \infty$, and $\log n/\sqrt{r_n} \to 0$. Then on a sufficiently rich probability space there exist versions of the X_k and Y_k, and independent sequences $\mathbf{B}_{1,n}$*

and $\mathbf{B}_{2,n}$ *of standard Brownian bridges on* $[0,1]$ *such that*

$$\big\|\mathbf{F}_n - \mathbf{B}_{1,l_n} \circ F\big\|_\infty = o_P\big(\sqrt{r_n/n}\big)\,, \quad \big\|\mathbf{G}_n - \mathbf{B}_{2,m_n} \circ G\big\|_\infty = o_P\big(\sqrt{r_n/n}\big)\,.$$

See del Barrio, Deheuvels and van de Geer (2007).

Lemma 5. *Let* l_n *be a sequence in* $\mathbb{N}$*,* $\mathbf{B}_n$ *be a sequence of standard Brownian bridges on* $[0,1]$*, and* X_n *be a random sequence (not necessarily independent of* $\mathbf{B}_n$*) such that* $X_n \to_p \gamma$ *for some* $\gamma \geq 0$*. For all real* $c_n \downarrow 0$ *it holds that*

$$\big|\mathbf{B}_{l_n}(c_n X_n) - \mathbf{B}_{l_n}(c_n\gamma)\big| = o_P(\sqrt{c_n})\,.$$

Proof. Let $Z_1, Z_2, \dots$ be a sequence of standard normal variables, and for $n \in \mathbb{N}$ and $t \in [0,1]$ define $\mathbf{W}_n(t) = \mathbf{B}_n(t) + tZ_n$. Then $\mathbf{W}_n$ is a sequence of standard Wiener processes, and

$$\begin{aligned}\mathbf{B}_{l_n}(c_n X_n) - \mathbf{B}_{l_n}(c_n\gamma) &= \mathbf{W}_{l_n}(c_n X_n) - \mathbf{W}_{l_n}(c_n\gamma) + c_n(\gamma - X_n)Z_{l_n} \\ &= \mathbf{W}_{l_n}(c_n X_n) - \mathbf{W}_{l_n}(c_n\gamma) + o_P(c_n)\,,\end{aligned}$$

so that the limit behavior under consideration de facto only depends on the properties of Brownian motion. For all $\varepsilon, \delta > 0$,

$$\begin{aligned}&\mathrm{P}\big(c_n^{-1/2}|\mathbf{W}_{l_n}(c_n X_n) - \mathbf{W}_{l_n}(c_n\gamma)| > \varepsilon\big) \\ &\quad\leq \mathrm{P}(|X_n - \gamma| > \delta) + \mathrm{P}\Big(\sup_{|t-\gamma|\leq\delta} c_n^{-1/2}|\mathbf{W}_{l_n}(c_n t) - \mathbf{W}_{l_n}(c_n\gamma)| > \varepsilon\Big) \\ &\quad= \mathrm{P}(|X_n - \gamma| > \delta) + \mathrm{P}\Big(\sup_{|t-\gamma|\leq\delta} |\mathbf{W}_1(t) - \mathbf{W}_1(\gamma)| > \varepsilon\Big)\end{aligned}$$

since, by Brownian scaling, each process $y \mapsto c_n^{-1/2}\mathbf{W}_{l_n}(c_n y)$ is itself a standard Brownian motion. Note also that, by the continuity of Brownian motion sample paths, the supremum has to be taken over $t \in \mathbb{Q}$ only, what makes it a measurable function. The left probability tends to zero for all $\delta > 0$ because $X_n \to_p \gamma$. The right probability can be made arbitrarily small by the choice of δ since, again by the almost sure continuity of $\mathbf{W}_1$,

$$\lim_{m\to\infty} \mathrm{P}\Big(\sup_{|t-\gamma|\leq 1/m} |\mathbf{W}_1(t) - \mathbf{W}_1(\gamma)| > \varepsilon\Big) = \mathrm{P}\Big(\bigcap_{m\in\mathbb{N}} \Big\{\sup_{|t-\gamma|\leq 1/m} |\mathbf{W}_1(t) - \mathbf{W}_1(\gamma)| > \varepsilon\Big\}\Big) = 0\,.$$

Conclude that $c_n^{-1/2}|\mathbf{B}_{l_n}(c_n X_n) - \mathbf{B}_{l_n}(c_n\gamma)| \to_p 0$. $\square$

Since $\mathbf{B}$ is zero mean Gaussian with covariance $\mathrm{E}\big(\mathbf{B}(s)\mathbf{B}(t)\big) = (s \wedge t) - st$, $s, t \in [0,1]$, it readily follows that

$$c_n^{-1/2}\mathbf{B}(c_n\gamma) \sim_d \mathcal{N}\big(0, \gamma(1 - c_n\gamma)\big) \longrightarrow \mathcal{N}(0,\gamma) \quad \text{as } n \to \infty\,. \tag{2.13}$$

Proof of (2.12). Write

$$\sqrt{r_n}\big(\theta_n - \bar\theta_n\big) = \frac{\sqrt{r_n/m_n}}{G_n(h_n)}\Big(\sqrt{m_n/l_n}\,\mathbf{F}_n(h_n) - \bar\theta_n\,\mathbf{G}_n(h_n)\Big), \tag{2.14}$$

where $\mathbf{F}_n = \sqrt{l_n}(F_n - F)$ and $\mathbf{G}_n = \sqrt{m_n}(G_n - G)$ denote the empirical processes. By Lemma 4 there exists a sequence $\mathbf{B}_{2,n}$ of standard Brownian bridges such that

$$\mathbf{G}_n(h_n) = \mathbf{B}_{2,m_n}(G(h_n)) + o_P\big(\sqrt{r_n/n}\big). \tag{2.15}$$

Now, (2.4) and (2.11) imply that

$$\frac{n}{r_n}\big|G(h_n) - r_n/m_n\big| \le \frac{n}{r_n}\|G - G_n\|_\infty + o_P(1) = \frac{\sqrt{n}}{r_n}O_P(1) + o_P(1) = o_P(1),$$

yielding $G(h_n) = r_n/m_n\big(1 + o_P(1)\big)$. Inserting this in (2.15) and using Lemma 5 (with $\gamma = 1$) we find that

$$\mathbf{G}_n(h_n) = \mathbf{B}_{2,m_n}(r_n/m_n) + o_P\big(\sqrt{r_n/n}\big).$$

Similarly for $\mathbf{F}_n(h_n)$, there is an independent sequence $\mathbf{B}_{1,n}$ of standard Brownian bridges such that, using $\bar\theta_n \to_p \theta$ and $F(h_n) = \bar\theta_n G(h_n) = r_n/m_n\big(\bar\theta_n + o_P(1)\big)$,

$$\mathbf{F}_n(h_n) = \mathbf{B}_{1,l_n}(\theta\, r_n/m_n) + o_P\big(\sqrt{r_n/n}\big).$$

Therefore, using (2.14) and (2.11),

$$\begin{aligned}\sqrt{r_n}\big(\theta_n - \bar\theta_n\big) &= \frac{\sqrt{r_n/m_n}}{G_n(h_n)}\Big(\sqrt{\frac{m_n}{l_n}}\mathbf{F}_n(h_n) - \bar\theta_n\mathbf{G}_n(h_n)\Big)\\ &= \frac{\sqrt{m_n/r_n}}{1 + o_P(1)}\Big(\sqrt{\tau}\,\mathbf{B}_{1,l_n}(\theta\, r_n/m_n) - \theta\,\mathbf{B}_{2,m_n}(r_n/m_n)\Big) + o_P(1)\,, \end{aligned}\tag{2.16}$$

so that the result follows from (2.13) and the independence of $\mathbf{B}_{1,n}$ and $\mathbf{B}_{2,n}$. □

Proof of asymptotic independence in Theorem 3. We say that sequences of random vectors X_n in $\mathbb{R}^p$ and Y_n in $\mathbb{R}^q$ are asymptotically independent if

$$\mathrm{E}\big(f(X_n)g(Y_n)\big) - \mathrm{E}\big(f(X_n)\big)\mathrm{E}\big(g(Y_n)\big) \longrightarrow 0 \quad \text{as } n \to \infty$$

for all bounded, non-negative, Lipschitz functions f and g on $\mathbb{R}^p$ and $\mathbb{R}^q$, resp. For the next lemma see Example 1.4.6 in van der Vaart and Wellner (2000).

Lemma 6. *If there exist independent random vectors X and Y such that $X_n \rightsquigarrow X$ and $Y_n \rightsquigarrow Y$, and if further X_n and Y_n are asymptotically independent, then $(X_n, Y_n)' \rightsquigarrow (X, Y)'$.*

The following lemma gives a criterion for asymptotic independence in case of Gaussian sequences, where boundedness and convergence of matrices is understood with respect to the Frobenius norm given by $\|M\|_F = \big(\sum_{i,j} |M_{i,j}|^2\big)^{1/2}$.

Lemma 7. *If X_n and Y_n are zero mean jointly Gaussian, the covariance matrices* $\mathrm{CoV}\, X_n$ *and* $\mathrm{CoV}\, Y_n$ *are uniformly bounded above and uniformly bounded away from zero, eventually, and if* $\mathrm{CoV}(X_n, Y_n) \to 0$*, then X_n and Y_n are asymptotically independent.*

Proof. Let f and g be positive, bounded, and Lipschitz. Denoting by $\phi_{(X_n,Y_n)}$ the joint density and by ϕ_{X_n} and ϕ_{Y_n} the marginal densities of X_n and Y_n,

$$\big|\mathrm{E}\big(f(X_n)g(Y_n)\big) - \mathrm{E}\big(f(X_n)\big)\mathrm{E}\big(g(Y_n)\big)\big| \le \iint f(x)g(y)\big|\phi_{(X_n,Y_n)}(x,y) - \phi_{X_n}(x)\phi_{Y_n}(y)\big|\, dx\, dy\,.$$

By the boundedness of $\mathrm{CoV}\, X_n$ and $\mathrm{CoV}\, Y_n$, and by the convergence $\mathrm{CoV}(X_n, Y_n) \to 0$,

$$|\phi_{(X_n,Y_n)}(x,y) - \phi_{X_n}(x)\phi_{Y_n}(y)| \longrightarrow 0\,, \quad x, y \in \mathbb{R}\,.$$

Hence, minding that the densities are uniformly bounded above by an integrable function due to the boundedness of $\mathrm{CoV}\, X_n$ and $\mathrm{CoV}\, Y_n$, the result follows in view of Lebesgue's dominated convergence. □

We are now ready to come back to the estimators θ_n and η_n. Regarding (2.16), (2.3) and Lemma 12 in Hohmann and Holzmann (2013), under certain assumptions there exist independent sequences $\mathbf{B}_{1,n}$ and $\mathbf{B}_{2,n}$ of standard Brownian bridges such that

$$\begin{aligned}
\sqrt{r_n}\big(\theta_n - \bar\theta_n\big) &= \frac{\sqrt{m_n/r_n}}{1 + o_P(1)}\Big(\sqrt{\tau}\,\mathbf{B}_{1,l_n}(\theta\, r_n/m_n) - \theta\,\mathbf{B}_{2,m_n}(r_n/m_n)\Big) + o_P(1),\\
\sqrt{r_n}\big(\eta_n - \bar\eta_n\big) &= \frac{z_n}{(1 + o_P(1))}\Big(\sqrt{\tau}\,\mathbf{C}_{1,l_n}(t_n) - \eta\mathbf{C}_{2,m_n}(t_n)\Big) + o_P(1),
\end{aligned}$$

where

$$\mathbf{C}_{1,n}(y) = \int \exp(iyx)\,\mathbf{B}_{1,n}(F(dx)), \quad \mathbf{C}_{2,n}(y) = \int \exp(iyx)\,\mathbf{B}_{2,n}(G(dx)).$$

Hence, it suffices to concentrate on the sequences

$$\begin{aligned}
A_n &= \sqrt{m_n/r_n}\Big(\sqrt{\tau}\,\mathbf{B}_{1,l_n}(\theta\, r_n/m_n) - \theta\,\mathbf{B}_{2,m_n}(r_n/m_n)\Big)\,,\\
B_n &= \begin{pmatrix} \mathrm{Re}\,\big(z_n\big(\sqrt{\tau}\,\mathbf{C}_{1,l_n}(t_n) - \eta\,\mathbf{C}_{2,m_n}(t_n)\big)\big) \\ \mathrm{Im}\,\big(z_n\big(\sqrt{\tau}\,\mathbf{C}_{1,l_n}(t_n) - \eta\,\mathbf{C}_{2,m_n}(t_n)\big)\big) \end{pmatrix}.
\end{aligned}$$

By construction of the stochastic integrals $\mathbf{C}_{k,n}(t)$, the vector $(A_n, B_n')'$ is zero mean trivariate Gaussian. In view of (2.13),

$$\mathrm{Var}\, A_n = \tau\theta(1 - \theta\, r_n/m_n) + \theta^2(1 - r_n/m_n)\,.$$

The variable $z_n\sqrt{\tau}\,\mathbf{C}_{1,l_n}(t_n) - z_n\eta\,\mathbf{C}_{2,m_n}(t_n)$ is complex Gaussian, having variance $\sigma_n^2 = \tau(1-|\widetilde{F}(t_n)|^2) + |\eta|^2(1-\widetilde{G}(t_n)|^2)$ and relation $\rho_n = z_n^2\tau(\widetilde{F}(2t_n) - \widetilde{F}(t_n)^2) + z_n^2\eta^2(\widetilde{G}(2t_n) - \widetilde{G}(t_n)^2)\big)$. Hence, both $\operatorname{Var} A_n$ and

$$\operatorname{CoV} B_n = \frac{1}{2}\begin{pmatrix} \sigma_n^2 + \operatorname{Re}\rho_n & \operatorname{Im}\rho_n \\ \operatorname{Im}\rho_n & \sigma_n^2 - \operatorname{Re}\rho_n \end{pmatrix}$$

are uniformly bounded above and uniformly bounded away from zero. For the asymptotic independence of A_n and B_n, it thus remains to show that $\operatorname{CoV}(A_n, B_n) \to 0$, which we will do exemplarily for A_n and B_n^1, the first coordinate of B_n.

First, note that $\mathbf{B}_{1,l_n}$ is independent of $\mathbf{C}_{2,m_n}$, and $\mathbf{B}_{2,m_n}$ is independent of $\mathbf{C}_{1,l_n}$. This yields

$$\begin{aligned}
\operatorname{CoV}(A_n, B_n^1) &= \sqrt{\frac{m_n}{r_n}}\Big(\tau\operatorname{CoV}\big(\mathbf{B}_{1,l_n}(\theta\, r_n/m_n), \operatorname{Re}(z_n\mathbf{C}_{1,l_n}(t_n))\big) \\
&\qquad + \theta\operatorname{CoV}\big(\mathbf{B}_{2,m_n}(r_n/m_n), \operatorname{Re}(\eta\mathbf{C}_{2,m_n}(t_n))\big)\Big) \\
&= \sqrt{\frac{m_n}{r_n}}\Big(\tau\operatorname{Re} z_n\operatorname{E}\big(\mathbf{B}_{1,l_n}(\theta\, r_n/m_n)\operatorname{Re}\mathbf{C}_{1,l_n}(t_n)\big) \\
&\qquad - \tau\operatorname{Im} z_n\operatorname{E}\big(\mathbf{B}_{1,l_n}(\theta\, r_n/m_n)\operatorname{Im}\mathbf{C}_{1,l_n}(t_n)\big) \\
&\qquad + \theta\operatorname{Re}\eta\operatorname{E}\big(\mathbf{B}_{2,m_n}(r_n/m_n)\operatorname{Re}\mathbf{C}_{2,m_n}(t_n)\big) \\
&\qquad - \theta\operatorname{Im}\eta\operatorname{E}\big(\mathbf{B}_{2,m_n}(r_n/m_n)\operatorname{Im}\mathbf{C}_{2,m_n}(t_n)\big)\Big)\,.
\end{aligned}$$

Hence, the last four expectations should be $o\big(\sqrt{r_n/m_n}\big)$. Exemplarily again, we only consider the first one. For convenience, set $\mathbf{B} = \mathbf{B}_{1,l_n}$, $\mathbf{C} = \mathbf{C}_{1,l_n}$, and let $\mathbf{W}$ be a standard Brownian motion on $[0,1]$, so that the processes $\mathbf{B}(t)$ and $\mathbf{W}(t) - t\mathbf{W}(1)$ are equal in distribution. With this,

$$\begin{aligned}
\operatorname{E}\big(\mathbf{B}(\theta\, r_n/m_n)\operatorname{Re}\mathbf{C}(t_n)\big) &= \operatorname{E}\Big(\big(\mathbf{W}(\theta\, r_n/m_n) - \frac{r_n}{m_n}\mathbf{W}(1)\big) \\
&\qquad \cdot\Big(\int_0^1 \cos(t_nF^{-1}(y))\,\mathbf{W}(dy) - \mathbf{W}(1)\int_0^1 \cos(t_nF^{-1}(y))\,dy\Big)\Big) \\
&= \int_0^{\theta\, r_n/m_n} \cos(t_nF^{-1}(y))\,dy - \frac{r_n}{m_n}\int_0^1 \cos(t_nF^{-1}(y))\,dy
\end{aligned}$$

which is in fact of the required order. Therefore, A_n and B_n are asymptotically independent, and in view of Lemma 6 we have proven Corollary 3. □

References

Csörgő, S. (1981). Limit behaviour of the empirical characteristic function. *Ann. Probab.*, **9** 130–144.

del Barrio, E., Deheuvels, P. and van de Geer, P. (2007). *Lectures on Empirical Processes: Theory and Statistical Applications.* EMS Series of Lectures in Mathematics, European Mathematical Society.

Hohmann, D. and Holzmann, H. (2013). Two-component mixtures with independent coordinates as conditional mixtures: Nonparametric identification and estimation. *Preprint.*

van der Vaart, A. and Wellner, J. (2000). *Weak Convergence and Empirical Processes.* Springer Series in Statistics, Springer, New York.