

# Weak convergence of quantile and expectile processes under general assumptions

Tobias Zwingmann\* and Hajo Holzmann†

Fachbereich Mathematik und Informatik  
Philipps-Universität Marburg, Germany

June 14, 2017

## Abstract

We show weak convergence of quantile and expectile processes to Gaussian limit processes in the space of bounded functions endowed with an appropriate semimetric which is based on the concepts of epi- and hypo convergence as introduced in Bücher et al. (2014). We impose assumptions for which it is known that weak convergence with respect to the supremum norm or the Skorodhod metric generally fails to hold. For expectiles, we only require a distribution with finite second moment but no further smoothness properties of distribution function, for quantiles, the distribution is assumed to be absolutely continuous with a version of its Lebesgue density which is strictly positive and has left- and right-sided limits. We also show consistency of the bootstrap for this mode of convergence.

*Keywords.* epi - and hypo convergence, expectile process, quantile process, weak convergence

## 1 Introduction

Quantiles and expectiles are fundamental parameters of a distribution which are of major interest in statistics, econometrics and finance (Koenker, 2005; Bellini et al., 2014; Newey and Powell, 1987; Ziegel, 2016).

The asymptotic properties of sample quantiles and expectiles have been addressed in detail under suitable conditions. For quantiles, differentiability of the distribution function at the quantile with positive derivative implies asymptotic normality of the empirical quantile, and under a continuity assumption on the density one obtains weak convergence of the quantile process to a Gaussian limit process in the space of bounded functions with the supremum distance from the functional delta-method (van der Vaart, 2000). However, without a positive derivative of the distribution function at the quantile, the weak limit will be non-normal (Knight, 2002), and thus process convergence to a Gaussian limit with respect to the supremum distance cannot hold true.

Similarly, for a distribution with finite second moment, the empirical expectile is asymptotically normally distributed if the distribution function is continuous at the expectile, but non-normally distributed otherwise (Holzmann and Klar, 2016). For continuous distribution functions, process convergence of the empirical expectile process in the space of continuous functions also holds true, but for discontinuous distribution functions this can no longer be valid.

---

\*Email: zwingmann@mathematik.uni-marburg.de

†Corresponding author, Email: holzmann@mathematik.uni-marburg.de

In this note we discuss convergence of quantile and expectile processes from independent and identically distributed observations under more general conditions. Indeed, we show that the expectile process converges to a Gaussian limit in the semimetric space of bounded functions endowed with the hypi-semimetric as recently introduced in Bücher et al. (2014) under the assumption of a finite second moment only. Since the Gaussian limit process is discontinuous in general while the empirical expectile process is continuous, this convergence can hold neither with respect to the supremum distance nor with a variant of the Skorodhod metric. Similarly, we show weak convergence of the quantile process under the hypi-semimetric if the distribution function is absolutely continuous with a version of its Lebesgue density that is strictly positive and has left and right limits at any point. These results still imply weak convergence of important statistics such as Kolmogorov-Smirnov and Cramer-von Mises type statistics. We also show consistency of the  $n$  out of  $n$  bootstrap in both situations.

A sequence of bounded functions  $(f_n)_{n \in \mathbb{N}}$  on a compact metric space hypi-converges to the bounded function  $f$  if it epi-converges to the lower-semicontinuous hull of  $f$ , and hypo-converges to its upper-semicontinuous hull, see the supplementary material for the definitions. Bücher et al. (2014) show that this mode of convergence can be expressed in terms of a semimetric  $d_{\text{hypi}}$  on the space of bounded functions, the hypi-semimetric.

We shall use the notation  $c^{-1} = 1/c$  for  $c \in \mathbb{R}$ ,  $c \neq 0$ , and denote the (pseudo-) inverse of a function  $h$  with  $h^{\text{Inv}}$ . We write  $\xrightarrow{\mathcal{L}}$  for ordinary weak convergence of real-valued random variables. Weak convergence in semimetric spaces will be understood in the sense of Hofmann-Jørgensen, see van der Vaart and Wellner (2013) and Bücher et al. (2014).

## 2 Convergence of the expectile process

For a random variable  $Y$  with distribution function  $F$  and finite mean  $\mathbb{E}[|Y|] < \infty$ , the  $\tau$ -expectile  $\mu_\tau = \mu_\tau(F)$ ,  $\tau \in (0, 1)$ , can be defined as the unique solution of  $\mathbb{E}[I_\tau(x, Y)] = 0$ ,  $x \in \mathbb{R}$ , where

$$I_\tau(x, y) = \tau(y - x)\mathbb{1}(y \geq x) - (1 - \tau)(x - y)\mathbb{1}(y < x), \quad (1)$$

and  $\mathbb{1}(\cdot)$  is the indicator function. Given a sequence of independent and identically distributed copies  $Y_1, Y_2, \dots$  of  $Y$  and a natural number  $n \in \mathbb{N}$ , we let

$$\hat{\mu}_{\tau,n} = \mu_\tau(F_n), \quad F_n(x) = \frac{1}{n} \sum_{k=1}^n \mathbb{1}(Y_k \leq x),$$

be the empirical  $\tau$ -expectile and the empirical distribution function, respectively.

**Theorem 1.** *Suppose that  $\mathbb{E}[Y^2] < \infty$ . Given  $0 < \tau_l < \tau_u < 1$ , the standardized expectile process  $\tau \mapsto \sqrt{n}(\hat{\mu}_{\tau,n} - \mu_\tau)$ ,  $\tau \in [\tau_l, \tau_u]$ , converges weakly in  $(\ell^\infty[\tau_l, \tau_u], d_{\text{hypi}})$  to the limit process  $(\dot{\psi}^{\text{Inv}}(Z)(\tau))_{\tau \in [\tau_l, \tau_u]}$ . Here,  $\dot{\psi}^{\text{Inv}}(\varphi)(\tau) = (\tau + (1 - 2\tau)F(\mu_\tau))^{-1}\varphi(\tau)$ ,  $\varphi \in \ell^\infty[\tau_l, \tau_u]$ , and  $(Z_\tau)_{\tau \in [\tau_l, \tau_u]}$  is a centered tight Gaussian process with continuous sample paths and covariance function  $\text{cov}(Z_\tau, Z_{\tau'}) = \mathbb{E}[I_\tau(\mu_\tau, Y)I_{\tau'}(\mu_{\tau'}, Y)]$  for  $\tau, \tau' \in [\tau_l, \tau_u]$ .*

From Propositions 2.3 and 2.4 in Bücher et al. (2014), hypi-convergence of the expectile process implies ordinary weak convergence of important statistics such as Kolmogorov-Smirnov or Cramér-von Mises type statistics. We let

$$\|\varphi\|_{[\tau_l, \tau_u]} = \sup_{\tau \in [\tau_l, \tau_u]} |\varphi(\tau)|, \quad \varphi \in \ell^\infty[\tau_l, \tau_u],$$

denote the supremum norm on  $\ell^\infty[\tau_l, \tau_u]$ .

**Corollary 2.** *If  $\mathbb{E}[Y^2] < \infty$ , then we have as  $n \rightarrow \infty$  that*

$$\sqrt{n} \|\hat{\mu}_{\cdot, n} - \mu_{\cdot}\|_{[\tau_l, \tau_u]} \xrightarrow{\mathcal{L}} \|\dot{\psi}^{\text{Inv}}(Z)\|_{[\tau_l, \tau_u]}.$$

Further, for  $p \geq 1$  and a bounded, non-negative weight function  $w$  on  $[\tau_l, \tau_u]$ ,

$$n^{p/2} \int_{\tau_l}^{\tau_u} |(\hat{\mu}_{\tau, n} - \mu_\tau)|^p w(\tau) d\tau \xrightarrow{\mathcal{L}} \int_{\tau_l}^{\tau_u} |(\dot{\psi}^{\text{Inv}}(Z))(\tau)|^p w(\tau) d\tau.$$

**Remark 3.** Evaluation at a given point  $x$  is only a continuous operation under the hypi-semi-metric if the limit function is continuous at  $x$ , see Proposition 2.2 in Bücher et al. (2014). In particular, this does not apply to the expectile process if the distribution function  $F$  is discontinuous at  $\mu_\tau$ . Indeed, Theorem 7 in Holzmann and Klar (2016) shows that the weak limit of the empirical expectile is not normal in this case.

Next we turn to the validity of the bootstrap. Given  $n \in \mathbb{N}$  let  $Y_1^*, \dots, Y_n^*$  denote a sample drawn from  $Y_1, \dots, Y_n$  with replacement, that is, having distribution function  $F_n$ . Let  $F_n^*$  denote the empirical distribution function of  $Y_1^*, \dots, Y_n^*$ , and let  $\mu_{\tau, n}^* = \mu_\tau(F_n^*)$  denote the bootstrap expectile at level  $\tau \in (0, 1)$ .

**Theorem 4.** *Suppose that  $\mathbb{E}[Y^2] < \infty$ . Then, almost surely, conditionally on  $Y_1, Y_2, \dots$  the standardized bootstrap expectile process  $\tau \mapsto \sqrt{n}(\mu_{\tau, n}^* - \hat{\mu}_{\tau, n})$ ,  $\tau \in [\tau_l, \tau_u]$ , converges weakly in  $(L^\infty[\tau_l, \tau_u], d_{\text{hypi}})$  to  $(\dot{\psi}^{\text{Inv}}(Z)(\tau))_{\tau \in [\tau_l, \tau_u]}$ , where the map  $\dot{\psi}^{\text{Inv}}$  and the process  $(Z_\tau)_{\tau \in [\tau_l, \tau_u]}$  are as in Theorem 1.*

**Remark.** The simple  $n$  out of  $n$  bootstrap does not apply for the empirical expectile at level  $\tau$  if  $F$  is discontinuous at  $\mu_\tau$ , see Knight (1998) for a closely related result for the quantile. Thus, Theorem 4 is somewhat surprising, but its conclusion is reasonable in view of Remark 3.

### 3 Convergence of the quantile process

Let  $q_\alpha = q_\alpha(F) = \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}$ ,  $\alpha \in (0, 1)$ , denote the  $\alpha$ -quantile of the distribution function  $F$ , and let  $\hat{q}_\alpha = q_\alpha(F_n)$  be the empirical  $\alpha$ -quantile of the sample  $Y_1, \dots, Y_n$ .

**Theorem 5.** *Suppose that the distribution function  $F$  is absolutely continuous and strictly increasing on  $[q_{\alpha_l}, q_{\alpha_u}]$ ,  $0 < \alpha_l < \alpha_u < 1$ . Assume that there is a version  $f$  of its density which is bounded and bounded away from zero on  $[q_{\alpha_l}, q_{\alpha_u}]$ , and which admits right- and left-sided limits in every point of  $[q_{\alpha_l}, q_{\alpha_u}]$ .*

*Then the standardized quantile process  $\alpha \mapsto \sqrt{n}(\hat{q}_{\alpha, n} - q_\alpha)$ ,  $\alpha \in [\alpha_l, \alpha_u]$ , converges weakly in  $(L^\infty[\alpha_l, \alpha_u], d_{\text{hypi}})$  to the process  $(V_\alpha/f(q_\alpha))_{\alpha \in [\alpha_l, \alpha_u]}$ , where  $(V_\alpha)_{\alpha \in [\alpha_l, \alpha_u]}$  is a Brownian bridge on  $[\alpha_l, \alpha_u]$ .*

Furthermore, as  $n \rightarrow \infty$  we have that

$$\sqrt{n} \|\hat{q}_{\cdot, n} - q_{\cdot}\|_{[\alpha_l, \alpha_u]} \xrightarrow{\mathcal{L}} \|\dot{\Psi}^{\text{Inv}}(V)\|_{[\alpha_l, \alpha_u]},$$

as well as

$$n^{p/2} \int_{\alpha_l}^{\alpha_u} |\hat{q}_{\alpha, n} - q_\alpha|^p w(\alpha) d\alpha \xrightarrow{\mathcal{L}} \int_{\alpha_l}^{\alpha_u} |(\dot{\Psi}^{\text{Inv}}(V))(\alpha)|^p w(\alpha) d\alpha.$$

for  $p \geq 1$  and a bounded, non-negative weight function  $w$  on  $[\alpha_l, \alpha_u]$ .

Let  $q_{\alpha,n}^* = q_\alpha(F_n^*)$  denote the bootstrap quantile at level  $\alpha \in (0, 1)$ .

**Theorem 6.** *Let the assumptions of Theorem 5 be true. Then, almost surely, conditionally on  $Y_1, Y_2, \dots$  the standardized bootstrap quantile process  $\alpha \mapsto \sqrt{n}(q_{\alpha,n}^* - \hat{q}_{\alpha,n})$ ,  $\alpha \in [\alpha_l, \alpha_u]$ , converges weakly in  $(L^\infty[\alpha_l, \alpha_u], d_{\text{hypi}})$  to  $(V_\alpha/f(q_\alpha))_{\alpha \in [\alpha_l, \alpha_u]}$ .*

## 4 Appendix: Outlines of the proofs

In this section we present an outline of the proofs, technical details can be found in the supplementary material.

*Proof of Theorem 1 (Outline).* We give an outline of the proof of Theorem 1. For a distribution function  $S$  with finite first moment let  $[\psi(\varphi, S)](\tau) = -I_\tau(\varphi(\tau), S)$ ,  $\tau \in [\tau_l, \tau_u]$  and  $\varphi \in \ell^\infty[\tau_l, \tau_u]$ , where  $I_\tau(x, S) = \int I_\tau(x, y) dS(y)$ , and set  $\psi_0 = \psi(\cdot, F)$  and  $\psi_n = \psi(\cdot, \hat{F}_n)$ . In the following we simply write  $\|\varphi\|$  instead of  $\|\varphi\|_{[\tau_l, \tau_u]}$ .

*Step 1. Weak convergence of  $\sqrt{n}(\psi_0(\hat{\mu}_{\cdot,n}) - \psi_0(\mu_{\cdot}))$  to  $Z$  in  $(\ell^\infty[\tau_l, \tau_u], \|\cdot\|)$*

This step uses standard results from empirical process theory based on bracketing properties of Lipschitz-continuous functions. The main issue in the proof of the lemma below is the Lipschitz-continuity of  $\tau \mapsto \mu_\tau$ ,  $\tau \in [\tau_l, \tau_u]$ , for a general distribution function  $F$ .

**Lemma 7.** *In  $(\ell^\infty[\tau_l, \tau_u], \|\cdot\|)$  we have the weak convergence*

$$\sqrt{n}(\psi_n(\mu_{\cdot}) - \psi_0(\mu_{\cdot}))(\tau) \rightarrow Z_\tau, \quad \tau \in [\tau_l, \tau_u]. \quad (2)$$

*Further, given  $\delta_n \searrow 0$  we have as  $n \rightarrow \infty$  that*

$$\begin{aligned} \sup_{\|\varphi\|_{[\tau_l, \tau_u]} \leq \delta_n} \sup_{\tau \in [\tau_l, \tau_u]} \sqrt{n} |\psi_n(\mu_{\cdot} + \varphi)(\tau) - \psi_0(\mu_{\cdot} + \varphi)(\tau) \\ - [\psi_n(\mu_{\cdot})(\tau) - \psi_0(\mu_{\cdot})(\tau)]| = o_P(1). \end{aligned} \quad (3)$$

Since  $\psi_0(\mu_{\cdot}) = \psi_n(\hat{\mu}_{\cdot,n}) = 0$ , (3) and the uniform consistency of  $\hat{\mu}_{\cdot,n}$  from Theorem 1 in Holzmann and Klar (2016) give

$$\begin{aligned} \sqrt{n}(\psi_0(\hat{\mu}_{\cdot,n}) - \psi_0(\mu_{\cdot})) &= \sqrt{n}(\psi_0(\hat{\mu}_{\cdot,n}) - \psi_n(\hat{\mu}_{\cdot,n})) \\ &= -\sqrt{n}(\psi_n(\mu_{\cdot}) - \psi_0(\mu_{\cdot})) + o_P(1), \end{aligned}$$

where the remainder term is in  $(\ell^\infty[\tau_l, \tau_u], \|\cdot\|)$ , and (2) and the fact that  $Z$  and  $-Z$  have the same law conclude the proof of

$$\sqrt{n}(\psi_0(\hat{\mu}_{\cdot,n}) - \psi_0(\mu_{\cdot})) \rightarrow Z \quad \text{weakly in } (\ell^\infty[\tau_l, \tau_u], \|\cdot\|). \quad (4)$$

*Step 2. Invertibility of  $\psi_0$  and semi-Hadamard differentiability of  $\psi_0^{\text{Inv}}$  with respect to  $d_{\text{hypi}}$ .*

The first part of Step 2. is observing the following lemma.

**Lemma 8.** *The map  $\psi_0$  is invertible. Further  $\psi_0^{\text{Inv}}(\varphi) \in \ell^\infty[\tau_l, \tau_u]$  for any  $\varphi \in \ell^\infty[\tau_l, \tau_u]$ .*

The next result then is the key technical ingredient in the proof of Theorem 1.

**Lemma 9.** *The map  $\psi_0^{\text{Inv}}$  is semi-Hadamard differentiable with respect to the hypi-semimetric in  $0 \in \mathcal{C}[\tau_l, \tau_u]$  tangentially to  $\mathcal{C}[\tau_l, \tau_u]$  with semi-Hadamard derivative  $\dot{\psi}^{\text{Inv}}(\varphi)(\tau) = (\tau + (1 - 2\tau)F(\mu_\tau))^{-1}\varphi(\tau)$ ,  $\varphi \in \ell^\infty[\tau_l, \tau_u]$ , that is, we have*

$$t_n^{-1}(\psi_0^{\text{Inv}}(t_n \varphi_n) - \psi_0^{\text{Inv}}(0)) \rightarrow \dot{\psi}^{\text{Inv}}(\varphi)(\tau)$$

for any sequence  $t_n \searrow 0$ ,  $t_n > 0$  and  $\varphi_n \in \ell^\infty[\tau_l, \tau_u]$  with  $\varphi_n \rightarrow \varphi \in \mathcal{C}[\tau_l, \tau_u]$  with respect to  $d_{\text{hypi}}$ .

The proof of the lemma is based on an explicit representation of increments of  $\psi_0^{\text{Inv}}$ , and novel technical properties of convergence under the hypi-semimetric for products and quotients.

*Step 3. Conclusion with the generalized functional delta-method.*

From (4), Lemma 9 and the generalized functional delta-method, Theorem B.7 in Bücher et al. (2014) we obtain

$$\sqrt{n}(\hat{\mu}_{\cdot, n} - \mu_{\cdot}) = \sqrt{n}(\psi_0^{\text{Inv}}(\psi_0(\hat{\mu}_{\cdot, n})) - \psi_0^{\text{Inv}}(0)) \rightarrow \dot{\psi}^{\text{Inv}}(Z)$$

in  $(L^\infty[\tau_l, \tau_u], d_{\text{hypi}})$ . □

*Proof of Theorem 4 (Outline).* The steps in the proof are similar to those of Theorem 1. In the analogous result to Lemma 7 and (4), we require the uniform consistency of  $\mu_{\cdot, n}^*$  as in Holzmann and Klar (2016), Theorem 1. The weak convergence statements require the changing classes central limit theorem, van der Vaart (2000), Theorem 19.28. In the second step we argue directly with the extended continuous mapping theorem, Theorem B.3 in Bücher et al. (2014). □

*Proof of Theorem 5 (Outline).* Let  $G$  be the distribution function of  $\mathcal{U}(0, 1)$  and  $G_n$  the empirical version coming from a sample  $U_1, \dots, U_n$ . By the quantile transformation we can write

$$\sqrt{n}(\hat{q}_{\cdot, n} - q_{\cdot}) = \sqrt{n}(F^{\text{Inv}}(G_n^{\text{Inv}}(\cdot)) - F^{\text{Inv}}(G^{\text{Inv}}(\cdot))).$$

The process  $\sqrt{n}(G_n^{\text{Inv}} - G^{\text{Inv}})$  converges in distribution in  $(l^\infty[\alpha_l, \alpha_u], \|\cdot\|_{[\alpha_l, \alpha_u]})$  to  $V$ , see Example 19.6, van der Vaart (2000).

To use a functional delta-method for the hypi-semimetric, we require

$$t_n^{-1}(F^{\text{Inv}}(\alpha + t_n \varphi_n(\alpha)) - F^{\text{Inv}}(\alpha)) \rightarrow \frac{\varphi(\alpha)}{f(q_\alpha)}$$

with respect to  $d_{\text{hypi}}$  for  $t_n \searrow 0$  and  $\varphi_n \in \ell^\infty[\alpha_l, \alpha_u]$ ,  $\varphi \in \mathcal{C}[\alpha_l, \alpha_u]$  such that  $d_{\text{hypi}}(\varphi_n, \varphi) \rightarrow 0$ , that is the semi-Hadamard-differentiability of the functional  $\Psi_0^{\text{Inv}}(\nu)(\alpha) = F^{\text{Inv}}(\alpha + \nu(\alpha))$ ,  $\nu \in \ell^\infty[\alpha_l, \alpha_u]$  as in Definition B.6, Bücher et al. (2014). To also be able to deal with the bootstrap version, we directly show a slightly stronger version, the *uniform* semi-Hadamard-differentiability.

**Lemma 10.** *Let  $t_n \searrow 0 \in \mathbb{R}$ ,  $\varphi_n, \nu_n \in \ell^\infty[\alpha_l, \alpha_u]$ ,  $\varphi \in \mathcal{C}[\alpha_l, \alpha_u]$  such that  $d_{\text{hypi}}(\varphi_n, \varphi) \rightarrow 0$  and  $d_{\text{hypi}}(\nu_n, 0) \rightarrow 0$  holds. Then the convergence*

$$t_n^{-1} \left[ F^{\text{Inv}} \{ id_{[\alpha_l, \alpha_u]}(\cdot) + \nu_n(\cdot) + t_n \varphi_n(\cdot) \} - F^{\text{Inv}} \{ id_{[\alpha_l, \alpha_u]}(\cdot) + \nu_n(\cdot) \} \right] \rightarrow \frac{\varphi(\cdot)}{f(q)}$$

with respect to  $d_{\text{hypi}}$  is true.

Using the convergence of  $\sqrt{n}(G_n^{\text{Inv}} - G^{\text{Inv}})$ , Lemma 10 and the functional delta-method, Theorem B.7, Bücher et al. (2014), we conclude

$$\sqrt{n}(\hat{q}_{\cdot,n} - q) = \sqrt{n} \left[ F^{\text{Inv}} \{ G_n^{\text{Inv}}(\cdot) \} - F^{\text{Inv}} \{ G^{\text{Inv}}(\cdot) \} \right] \rightarrow \frac{V}{f(q)}$$

in  $(L^\infty[\alpha_l, \alpha_u], d_{\text{hyppi}})$ . □

*Proof of Theorem 6.* Given a sample  $U_1, \dots, U_n$  of independent  $\mathcal{U}[0, 1]$  distributed random variables and the corresponding empirical distribution function  $G_n$ , let  $U_1^*, \dots, U_n^*$  be a bootstrap sample with empirical distribution function  $G_n^*$ . Using the quantile transformation we obtain

$$\sqrt{n}(q_{\cdot,n}^* - \hat{q}_{\cdot,n}) = \sqrt{n} \left[ F^{\text{Inv}} \{ (G_n^*)^{\text{Inv}}(\cdot) \} - F^{\text{Inv}} \{ G_n^{\text{Inv}}(\cdot) \} \right].$$

By Theorem 3.6.2, van der Vaart and Wellner (2013), the process  $\sqrt{n}\{(G_n^*)^{\text{Inv}}(\cdot) - G_n^{\text{Inv}}(\cdot)\}$  converges in distribution to  $V$ , conditionally, almost surely, with  $V$  as in Theorem 5. Now we use Theorem 3.9.13, van der Vaart and Wellner (2013), extended to semi-metric spaces. To this end we use Lemma 10, which covers (3.9.12) in van der Vaart and Wellner (2013). Moreover we require that  $\|G_n^{\text{Inv}}(\cdot) - G^{\text{Inv}}(\cdot)\| \rightarrow 0$  holds almost surely, which is valid by the classical Glivenko-Cantelli-Theorem, Theorem 19.4, van der Vaart (2000). Finally, condition (3.9.9) in van der Vaart and Wellner (2013) is valid by their Theorem 3.6.2. Thus we can indeed apply Theorem 3.9.13, van der Vaart and Wellner (2013), which concludes the proof. □

## Supplementary material

The supplementary material contains simulation results, a summary of convergence under the hypi-semimetric and technical details for the proofs.

## Acknowledgements

Tobias Zwingmann acknowledges financial support from of the Cusanuswerk for providing a dissertation scholarship.

## References

- Bellini, F., B. Klar, A. Müller, and E. R. Gianin (2014). Generalized quantiles as risk measures. *Insur. Math. Econ.* 54, 41–48.
- Beyn, W.-J. and J. Rieger (2011). An implicit function theorem for one-sided lipschitz mappings. *Set-Valued and Variational Analysis* 19(3), 343–359.
- Bücher, A., J. Segers, and S. Volgushev (2014). When uniform weak convergence fails: empirical processes for dependence functions and residuals via epi- and hypographs. *Ann. Stat.* 42(4), 1598–1634.

- Holzmann, H. and B. Klar (2016). Expectile asymptotics. *Electron. J. Statist.* 10(2), 2355–2371.
- Knight, K. (1998). Bootstrapping sample quantiles in non-regular cases. *Stat. Probab. Lett.* 37, 259–267.
- Knight, K. (2002). What are the limiting distributions of quantile estimators? In Y. Dodge (Ed.), *Statistical Data Analysis Based on the L1-Norm and Related Methods*, pp. 47–65. Birkhäuser Basel.
- Koenker, R. (2005). *Quantile regression*. Cambridge: Cambridge University Press.
- Newey, W. and J. Powell (1987). Asymmetric least squares estimation and testing. *Econometrica* 55(4), 819–847.
- Thomson, B., A. Bruckner, and J. Bruckner (2008). *Real Analysis*. [www.classicalrealanalysis.com](http://www.classicalrealanalysis.com).
- van der Vaart, A. (2000). *Asymptotic Statistics*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press.
- van der Vaart, A. and J. Wellner (2013). *Weak Convergence and Empirical Processes: With Applications to Statistics*. Springer Series in Statistics. Springer New York.
- Ziegel, J. F. (2016). Coherence and elicibility. *Math. Finance* 26(4), 901–918.

## 5 Supplement: Simulation Study

In this section we illustrate the asymptotic results for the expectile process in a short simulation. Let  $Y$  be a random variable with distribution function

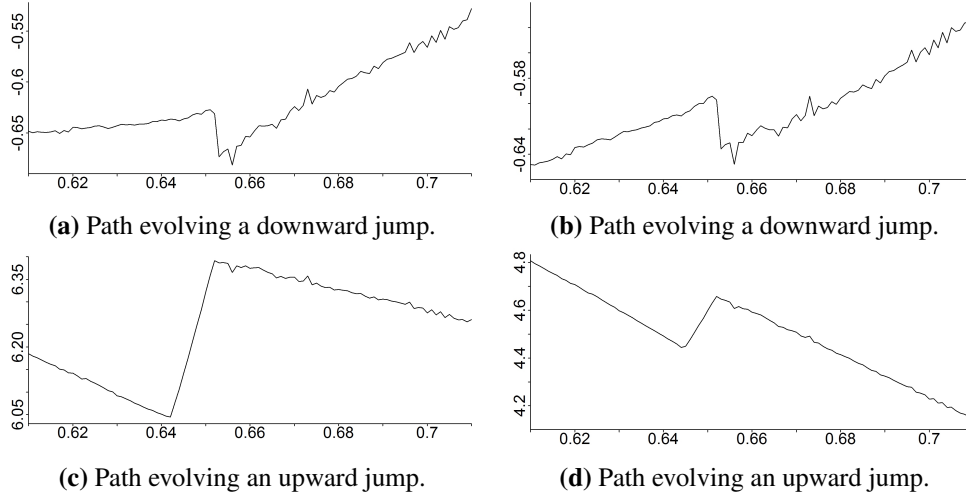
$$F(x) = \frac{9}{10} \int_{-\infty}^x \frac{1}{4\sqrt{2}\pi} \exp\left(-\frac{y^2}{32}\right) dy + \frac{1}{10} \mathbb{1}(x \geq 1),$$

which is a mixture of a  $\mathcal{N}(0, 16)$  random variable and point-mass in 1, so that  $\mathbb{E}[Y] = 1/10$  and  $\mathbb{E}[Y^2] = 14.5$ . We will concentrate on the weak convergence of the sup-norm of the empirical expectile process. Using equation (2.7) in Newey and Powell (1987), we numerically find  $\mu_{\tau_0} = 1$  for  $\tau_0 \approx 0.6529449$ , and investigate the expectile process on the interval  $[0.6, 0.7]$ .

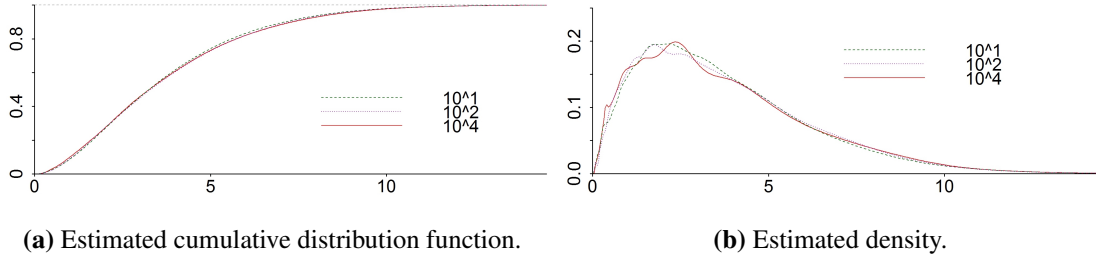
Figure 1 contains four paths of the expectile process  $\sqrt{n}(\hat{\mu}_{\tau,n} - \mu_{\tau})$  for samples of size  $n = 10^4$ . All plotted paths seem to evolve a jump around  $\tau_0$ .

Now we investigate the distribution of the supremum norm of the expectile process on the interval  $[0.6, 0.7]$ . To this end, we simulate  $M = 10^4$  samples of sizes  $n \in \{10, 10^2, 10^4\}$ , compute the expectile process and its supremum norm. Plots of the resulting empirical distribution function and density estimate of this statistic are contained in Figure 2. The distribution of the supremum distance seems to converge quickly.

Finally, to illustrate performance of the bootstrap, Figure 3 displays the distribution of  $M = 10^4$  bootstrap samples of  $\|\sqrt{n}(\mu_{\tau,n}^* - \hat{\mu}_{\tau,n})\|$  based on a single sample of size  $n \in \{10^2, 10^3, 10^4\}$  from the  $n$  out of  $n$  bootstrap, together with the distribution of  $\|\sqrt{n}(\hat{\mu}_{\tau,n} - \mu_{\tau})\|$ . The bootstrap distribution for  $n = 10^4$  is quite close to the empirical distribution.



**Figure 1:** The pictures show simulated paths of the empirical expectile process based on  $n = 10^4$  observations of  $Y$ . If the path is negative (positive) around  $\tau_0$ , a downward- (upward-) jump seems to evolve. This is plausible when considering the form of the hulls of  $\psi^{\text{Inv}}$  in the limit process.



**Figure 2:** Figure (a) shows the cumulative distribution function of the supremum norm of  $\sqrt{n}(\hat{\mu}_{\tau,n} - \mu_{\tau})$ , based on  $M = 10^4$  samples of sizes  $n \in \{10, 10^2, 10^4\}$ , Figure (b) the corresponding density estimate.

## 6 Supplement: weak-convergence under the hypi semimetric

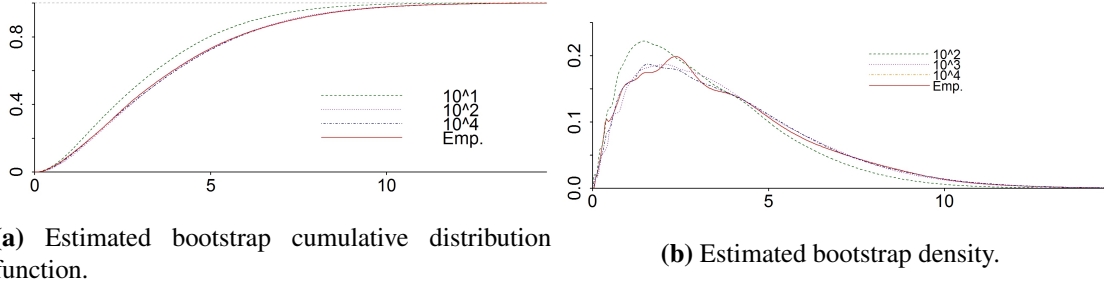
Let us briefly discuss the concept of hypi-convergence as introduced by Bücher et al. (2014). Let  $(\mathbb{T}, d)$  be a compact, separable metric space, and denote by  $\ell^\infty(\mathbb{T})$  the space of all bounded functions  $h : \mathbb{T} \rightarrow \mathbb{R}$ . The lower- and upper-semicontinuous hulls of  $h \in \ell^\infty(\mathbb{T})$  are defined by

$$h_\wedge(t) = \lim_{\varepsilon \searrow 0} \inf \{h(t') \mid d(t, t') < \varepsilon\}, \quad h_\vee(t) = \lim_{\varepsilon \nearrow 0} \sup \{h(t') \mid d(t, t') < \varepsilon\} \quad (5)$$

and satisfy  $h_\wedge, h_\vee \in \ell^\infty(\mathbb{T})$  as well as  $h_\wedge \leq h \leq h_\vee$ . A sequence  $h_n \in \ell^\infty(\mathbb{T})$  hypi-converges to a limit  $h \in \ell^\infty(\mathbb{T})$ , if it epi-converges to  $h_\wedge$ , that is,

$$\begin{aligned} &\text{for all } t, t_n \in \mathbb{T} \text{ with } t_n \rightarrow t : h_\wedge(t) \leq \liminf_{n \rightarrow \infty} h_n(t_n) \\ &\text{for all } t \in \mathbb{T} \text{ there exist } t_n \in \mathbb{T}, t_n \rightarrow t : h_\wedge(t) = \lim_{n \rightarrow \infty} h_n(t_n), \end{aligned} \quad (6)$$





**Figure 3:** Figure (a) shows the estimated cumulative bootstrap distribution function of  $\|\sqrt{n}(\hat{\mu}_{\cdot,n}^* - \mu_{\cdot,n})\|$ , figure (b) the estimated density thereof, obtained from  $M = 10^4$  estimates of this statistic. The red line indicates the estimated empirical distribution and density function, respectively, taken from  $\|\sqrt{n}(\mu_{\cdot,n} - \mu_{\cdot})\|$  for  $n = 10^4$ .

and if it hypo-converges to  $h_{\vee}$ , that is,

$$\begin{aligned} \text{for all } t, t_n \in \mathbb{T} \text{ with } t_n \rightarrow t : \quad & \limsup_{n \rightarrow \infty} h_n(t_n) \leq h_{\vee}(t) \\ \text{for all } t \in \mathbb{T} \text{ there exist } t_n \in \mathbb{T}, t_n \rightarrow t : \quad & \lim_{n \rightarrow \infty} h_n(t_n) = h_{\vee}(t). \end{aligned} \quad (7)$$

The limit function  $h$  is only determined in terms of its lower - and upper semicontinuous hulls. Indeed, there is a semimetric, denoted by  $d_{\text{hypi}}$ , so that the convergence in (5) and (7) is equivalent to  $d_{\text{hypi}}(h_n, h) \rightarrow 0$ , see Bücher et al. (2014) for further details. To transfer the concept of weak convergence from metric to semimetric space, Bücher et al. (2014) consider the space  $L^{\infty}(\mathbb{T})$  of equivalence classes  $[h] = \{g \in \ell^{\infty}(\mathbb{T}) \mid d_{\text{hypi}}(h, g) = 0\}$ . The convergence of a sequence of random elements  $(Y_n)$  in  $g \in \ell^{\infty}(\mathbb{T})$  to a Borel-measurable  $Y$  is defined in terms of weak convergence of  $([Y_n])$  to  $[Y]$  in the metric space  $(L^{\infty}(\mathbb{T}), d_{\text{hypi}})$  in the sense of Hofmann-Jørgensen, see van der Vaart and Wellner (2013)) and Bücher et al. (2014).

## 7 Supplement: Technical Details

We write  $\mathbb{E}_n[h(Y)] = 1/n \sum_{k=1}^n h(Y_k)$ , and use the abbreviation

$$\|\sqrt{n}(F_n - F)\|_{\mathcal{G}} = \sup_{h \in \mathcal{G}} |\sqrt{n}[\mathbb{E}_n[h(Y)] - \mathbb{E}[h(Y)]]|$$

for a class of measurable functions  $\mathcal{G}$ .

Recall from Holzmann and Klar (2016) the identity

$$I_{\tau}(x, F) = \tau \int_x^{\infty} (1 - F(y)) dy - (1 - \tau) \int_{-\infty}^x F(y) dy. \quad (8)$$

### 7.1 Details for the Proof of Theorem 1

We start with some technical preliminaries.

**Lemma 11.** *We have that for  $x_1, x_2 \in \mathbb{R}$ ,*

$$I_{\tau}(x_1, F) - I_{\tau}(x_2, F) = (x_2 - x_1) \left[ \tau + (1 - 2\tau) \int_0^1 F(x_2 + s(x_1 - x_2)) ds \right]. \quad (9)$$

*Proof.* Define  $g(s) = I_\tau(x_2 + s(x_1 - x_2), F)$ ,  $s \in [0, 1]$ , so that  $I_\tau(x_1, F) - I_\tau(x_2, F) = g(1) - g(0)$ . The map  $g$  is continuous, in addition it is decreasing, if  $x_1 \geq x_2$ , and increasing otherwise. Hence it is of bounded variation and Theorem 7.23, Thomson et al. (2008), yields

$$g(1) - g(0) = \int_0^1 g'(s) ds + \mu_g(\{s \in [0, 1] \mid g'(s) = \pm\infty\}), \quad (10)$$

where  $\mu_g$  is the Lebesgue-Stieltjes signed measure associated with  $g$ . From Holzmann and Klar (2016), the right- and left-sided derivatives of  $I_\tau(x, F)$  are given by

$$\frac{\partial^+}{\partial x} I_\tau(x, F) = -(\tau + (1 - 2\tau)F(x)), \quad \frac{\partial^-}{\partial x} I_\tau(x, F) = -(\tau + (1 - 2\tau)F(x-)). \quad (11)$$

Both derivatives are bounded, such that  $\{s \in [0, 1] \mid g'(s) = \pm\infty\} = \emptyset$  in (10), and we obtain

$$\begin{aligned} I_\tau(x_1, F) - I_\tau(x_2, F) &= \int_0^1 g'(s) ds \\ &= (x_1 - x_2) \int_0^1 -[\tau + (1 - 2\tau)F(x_2 + s(x_1 - x_2))] ds. \end{aligned}$$

□

**Lemma 12.** *We have*

$$\min\{\tau_l, 1 - \tau_u\} \leq \tau + (1 - 2\tau)s \leq 3/2, \quad \tau \in [\tau_l, \tau_u], \quad s \in [0, 1]. \quad (12)$$

*Proof.* For the lower bound,

$$\tau + (1 - 2\tau)s \begin{cases} = 1/2, & \text{if } \tau = 1/2 \\ \geq \tau, & \text{if } \tau < 1/2 \\ \geq 1 - \tau, & \text{if } \tau > 1/2 \end{cases} \geq \min\{1/2, \tau_l, 1 - \tau_u\} = \min\{\tau_l, 1 - \tau_u\}.$$

The upper bound is proved similarly. □

Next, we discuss Lipschitz-properties of relevant maps.

**Lemma 13.** *For any  $x_1, x_2, y \in \mathbb{R}$  and  $\tau \in [\tau_l, \tau_u]$ ,*

$$|I_\tau(x_1, y) - I_\tau(x_2, y)| \leq |x_2 - x_1| \quad (13)$$

*Further, for any  $\tau, \tau' \in [\tau_l, \tau_u]$  and  $x, y \in \mathbb{R}$ ,*

$$|I_\tau(x, y) - I_{\tau'}(x, y)| \leq |\tau - \tau'| (|x| + |y|) \quad (14)$$

*Finally, the map  $\tau \mapsto \mu_\tau$ ,  $\tau \in [\tau_l, \tau_u]$ , is Lipschitz-continuous.*

*Proof.* To show (13), for  $x_1 \leq x_2$ ,

$$\begin{aligned} |I_\tau(x_1, y) - I_\tau(x_2, y)| &= |(x_2 - x_1) (\tau \mathbb{1}(y > x_1) + (1 - \tau) \mathbb{1}(y \leq x_2))| \\ &\leq |x_2 - x_1|. \end{aligned}$$

As for (14),

$$\begin{aligned} |I_\tau(x, y) - I_{\tau'}(x, y)| &= |\tau - \tau'| |(y - x) \mathbb{1}_{y \geq x} + (x - y) \mathbb{1}_{y < x}| \\ &\leq |\tau - \tau'| (|x| + |y|). \end{aligned}$$

For the Lipschitz-continuity of  $\mu_\cdot$ , we use Corollary 1 of Beyn and Rieger (2011) for the function  $x \mapsto I_\tau(x, F)$ ,  $x \in B_R(\mu_\tau)$ , for appropriately chosen  $R > 0$ , where  $B_R(\mu_\tau)$  is the open ball around  $\mu_\tau$  with radius  $R$ . We observe that

(i)  $x \mapsto I_\tau(x, F)$  is continuous for any  $\tau \in [\tau_l, \tau_u]$ , which is immediate from (13),

(ii)  $x \mapsto I_\tau(x, F)$  fulfils

$$(I_\tau(x_1, F) - I_\tau(x_2, F)) (x_1 - x_2) \leq -a (x_1 - x_2)^2$$

with  $a = \min\{\tau_l, 1 - \tau_u\} > 0$ . This is clear from (9) and (12).

Let  $\tau, \tau' \in [\tau_l, \tau_u]$ , and set  $z = I_\tau(\mu_{\tau'}, F)$ . Using (ii) above yields

$$\frac{1}{a} |z| = \frac{1}{a} |I_\tau(\mu_{\tau'}, F) - I_\tau(\mu_\tau, F)| \leq |\mu_\tau - \mu_{\tau'}| \leq \mu_{\tau_u} - \mu_{\tau_l}.$$

Choosing  $R = \mu_{\tau_u} - \mu_{\tau_l}$  gives  $[\mu_{\tau_l}, \mu_{\tau_u}] \subset B_R(\mu_\tau) \subset [\mu_{\tau_l} - R, \mu_{\tau_u} + R]$ ,  $\tau \in [\tau_l, \tau_u]$ , since  $\mu_\tau \in [\mu_{\tau_l}, \mu_{\tau_u}]$ , and Corollary 1, Beyn and Rieger (2011), now gives a  $\bar{x} \in B_R(\mu_\tau)$  with  $I_\tau(\bar{x}, F) = z$  and

$$|\mu_\tau - \bar{x}| \leq \frac{1}{a} |z|.$$

Since  $x \mapsto I_\tau(x, F)$  is strictly decreasing and  $[\mu_{\tau_l}, \mu_{\tau_u}] \subset B_R(\mu_\tau)$  as well as  $I_\tau(\bar{x}, F) = z = I_\tau(\mu_{\tau'}, F)$ , we obtain  $\bar{x} = \mu_{\tau'}$ . We conclude that

$$\begin{aligned} |\mu_\tau - \mu_{\tau'}| &\leq \frac{1}{a} |I_\tau(\mu_{\tau'}, F)| = \frac{1}{a} |\mathbb{E}[I_\tau(\mu_{\tau'}, Y) - I_{\tau'}(\mu_{\tau'}, Y)]| \\ &\leq |\tau - \tau'| \frac{|\mu_{\tau'}| + \mathbb{E}[|Y|]}{a} \leq |\tau - \tau'| \frac{|\mu_{\tau_u}| \vee |\mu_{\tau_l}| + \mathbb{E}[|Y|]}{a}, \end{aligned} \quad (15)$$

where we used (14). □

*Details for Step 1.*

*Proof of Lemma 7. Proof of (2).*

By Lemma 13 the function class

$$\mathcal{F} = \{y \mapsto -I_\tau(\mu_\tau, y) \mid \tau \in [\tau_l, \tau_u]\}$$

is Lipschitz-continuous in the parameter  $\tau$  for given  $y$ , and the Lipschitz constant (which depends on  $y$ ) is square-integrable under  $F$ . Indeed, the triangle inequality first gives

$$|I_\tau(\mu_\tau, y) - I_{\tau'}(\mu_{\tau'}, y)| \leq |I_\tau(\mu_\tau, y) - I_{\tau'}(\mu_\tau, y)| + |I_{\tau'}(\mu_\tau, y) - I_{\tau'}(\mu_{\tau'}, y)|.$$

Using (14) the first summand on the right fulfils

$$|I_\tau(\mu_\tau, y) - I_{\tau'}(\mu_\tau, y)| \leq |\tau - \tau'| (|\mu_{\tau_l}| \vee |\mu_{\tau_u}| + |y|),$$

and the second is bounded by

$$|I_{\tau'}(\mu_\tau, y) - I_{\tau'}(\mu_{\tau'}, y)| \leq |\mu_\tau - \mu_{\tau'}| \leq |\tau - \tau'| \frac{|\mu_{\tau_u}| \vee |\mu_{\tau_l}| + \mathbb{E}[|Y|]}{a},$$

utilizing (13) and (15). Thus

$$|I_\tau(\mu_\tau, y) - I_{\tau'}(\mu_{\tau'}, y)| \leq |\tau - \tau'| (C + |y|) \quad (16)$$

for some constant  $C \geq 1$ . By example 19.7 in combination with Theorem 19.5 in van der Vaart (2000),  $\mathcal{F}$  is a Donsker class, so that  $\sqrt{n}(\psi_n(\mu_\cdot) - \psi_0(\mu_\cdot))$  converges to the process  $Z$ . The

same reasoning as in Theorem 8, Holzmann and Klar (2016), then shows continuity of the sample paths of  $Z$  with respect to the Euclidean distance on  $[\tau_l, \tau_u]$ .

*Proof of (3).*

Setting

$$\mathcal{F}_{\delta_n} = \{y \mapsto I_\tau(\mu_\tau + x, y) - I_\tau(\mu_\tau, y) \mid |x| \leq \delta_n, \tau \in [\tau_l, \tau_u]\}$$

we estimate that

$$\sup_{\|\varphi\|_{[\tau_l, \tau_u]} \leq \delta_n} \sup_{\tau \in [\tau_l, \tau_u]} \sqrt{n} |\psi_n(\mu_\cdot + \varphi)(\tau) - \psi_0(\mu_\cdot + \varphi)(\tau) - [\psi_n(\mu_\cdot)(\tau) - \psi_0(\mu_\cdot)(\tau)]|$$

is smaller than  $\|\sqrt{n}(F_n - F)\|_{\mathcal{F}_{\delta_n}}$ . From the triangle inequality, for any  $\tau, \tau' \in [\tau_l, \tau_u]$  and  $x, x' \in [-\delta_1, \delta_1]$  we first obtain

$$\begin{aligned} & |I_\tau(\mu_\tau + x, y) - I_\tau(\mu_\tau, y) - (I_{\tau'}(\mu_{\tau'} + x', y) - I_{\tau'}(\mu_{\tau'}, y))| \\ & \leq |I_\tau(\mu_\tau + x, y) - I_{\tau'}(\mu_{\tau'} + x', y)| + |I_\tau(\mu_\tau, y) - I_{\tau'}(\mu_{\tau'}, y)|, \end{aligned}$$

where the second term was discussed above and the first can be handled likewise to conclude

$$|I_\tau(\mu_\tau + x, y) - I_{\tau'}(\mu_{\tau'} + x', y)| \leq (|\tau - \tau'| + |x - x'|) (C + \delta_1 + |y|) \quad (17)$$

with the same  $C$  as above. Hence

$$\begin{aligned} & |I_\tau(\mu_\tau + x, y) - I_\tau(\mu_\tau, y) - (I_{\tau'}(\mu_{\tau'} + x', y) - I_{\tau'}(\mu_{\tau'}, y))| \\ & \leq m(y) (|\tau - \tau'| + |x - x'|) \end{aligned}$$

with Lipschitz-constant

$$m(y) = 2C + \delta_1 + 2|y|,$$

which is square-integrable by assumption on  $F$ . By example 19.7 in van der Vaart (2000) the bracketing number  $N_{[\cdot]}(\epsilon, \mathcal{F}_{\delta_1}, L_2(F))$  of  $\mathcal{F}_{\delta_1}$  is of order  $\epsilon^{-2}$ , so that for the bracketing integral

$$J_{[\cdot]}(\epsilon_n, \mathcal{F}_{\delta_n}, L_2(F)) \leq J_{[\cdot]}(\epsilon_n, \mathcal{F}_{\delta_1}, L_2(F)) \rightarrow 0 \quad \text{as } \epsilon_n \rightarrow 0.$$

From (13), the class  $\mathcal{F}_{\delta_n}$  has envelope  $\delta_n$ , and hence using Corollary 19.35 in van der Vaart (2000), we obtain

$$\mathbb{E} [\|\sqrt{n}(F_n - F)\|_{\mathcal{F}_{\delta_n}}] \leq J_{[\cdot]}(\delta_n, \mathcal{F}_{\delta_n}, L_2(F)) \rightarrow 0. \quad (18)$$

An application of the Markov inequality ends the proof of (3).  $\square$

*Details for Step 2.*

*Proof of Lemma 8.* Given  $\tau \in [\tau_l, \tau_u]$ , by (9) and the lower bound in (12), the function  $x \mapsto I_\tau(x, F)$  is strictly decreasing, and its image is all of  $\mathbb{R}$ . Hence, for any  $z \in \mathbb{R}$  there is a unique  $x$  satisfying  $I_\tau(x, F) = z$ , which shows that  $\psi_0$  is invertible.

Next for fixed  $\varphi \in \ell^\infty[\tau_l, \tau_u]$  the preimage  $((I_\tau(\cdot, F))^\text{Inv}([-\|\varphi\|, \|\varphi\|]))$  is by monotonicity an interval  $[L_\tau, U_\tau]$ ,  $|L_\tau|, |U_\tau| < \infty$ . By (8),

$$I_\tau(x, F) = \tau \left\{ \int_x^\infty (1 - F(y)) dy + \int_{-\infty}^x F(y) dy \right\} - \int_{-\infty}^x F(y) dy,$$

thus the map  $\tau \mapsto I_\tau(x, F)$  is increasing, showing  $L_{\tau'} \leq L_\tau$  and  $U_{\tau'} \leq U_\tau$  for  $\tau \geq \tau'$ . Hence the solution of  $z = I_\tau(x, F)$  for  $z \in [-\|\varphi\|, \|\varphi\|]$  lies in  $[L_{\tau_l}, U_{\tau_u}]$ , which means that  $\psi_0^\text{Inv}(\varphi)$  is bounded.  $\square$

Before we turn to the proof of Lemma 9, we require the following technical assertions about  $\psi_0^{\text{Inv}}$ .

**Lemma 14.** *Given  $t > 0$  and  $\nu \in \ell^\infty[\tau_l, \tau_u]$ , we have that*

$$\begin{aligned} & t^{-1}(\psi_0^{\text{Inv}}(t\nu) - \psi_0^{\text{Inv}}(0))(\tau) \\ &= \nu(\tau) \left\{ \tau + (1 - 2\tau) \int_0^1 F(\mu_\tau + s(\psi_0^{\text{Inv}}(t\nu)(\tau) - \mu_\tau)) \, ds \right\}^{-1}. \end{aligned} \quad (19)$$

*In particular, if  $\nu_n \in \ell^\infty[\tau_l, \tau_u]$  with  $\|\nu_n\| \rightarrow 0$ , then  $\|\psi_0^{\text{Inv}}(\nu_n)(\cdot) - \mu\| \rightarrow 0$ , so that for any  $\tau \in [\tau_l, \tau_u]$  and  $\tau_n \rightarrow \tau$ ,*

$$\psi_0^{\text{Inv}}(\nu_n)(\tau_n) - \mu_{\tau_n} \rightarrow 0. \quad (20)$$

*Proof.* For the first statement, given  $\rho \in \ell^\infty[\tau_l, \tau_u]$  it follows from (9) that

$$\begin{aligned} \psi_0(\rho)(\tau) &= \psi_0(\mu + (\rho - \mu))(\tau) - \psi_0(\mu)(\tau) \\ &= -\left( I_\tau(\mu_\tau + (\rho(\tau) - \mu_\tau), F) - I_\tau(\mu_\tau, F) \right) \\ &= (\rho(\tau) - \mu_\tau) \left\{ \tau + (1 - 2\tau) \int_0^1 F(\mu_\tau + s(\rho(\tau) - \mu_\tau)) \, ds \right\}. \end{aligned}$$

The integral on the right hand side is bounded away from zero by (12) and thus choosing  $\rho = \psi_0^{\text{Inv}}(t\nu)$ , observing  $\mu_\tau = \psi_0^{\text{Inv}}(0)(\tau)$  and reorganising the above equation leads to

$$\begin{aligned} & t^{-1}(\psi_0^{\text{Inv}}(0 + t\nu) - \psi_0^{\text{Inv}}(0))(\tau) \\ &= t^{-1} \psi_0(\psi_0^{\text{Inv}}(t\nu))(\tau) \left\{ \tau + (1 - 2\tau) \int_0^1 F(\mu_\tau + s(\psi_0^{\text{Inv}}(t\nu)(\tau) - \mu_\tau)) \, ds \right\}^{-1} \\ &= \nu(\tau) \left\{ \tau + (1 - 2\tau) \int_0^1 F(\mu_\tau + s(\psi_0^{\text{Inv}}(t\nu)(\tau) - \mu_\tau)) \, ds \right\}^{-1}. \end{aligned} \quad (21)$$

In the sequel, given  $\varphi \in \ell^\infty[\tau_l, \tau_u]$  let us introduce the function

$$c_\varphi(\tau) = \tau + (1 - 2\tau) \int_0^1 F(\mu_\tau + s\varphi(\tau)) \, ds. \quad (22)$$

By (12),  $\min\{\tau_l, 1 - \tau_u\} \leq c_\varphi \leq 3/2$  holds uniformly for any  $\varphi$ .

Now, for the second part, set  $\varphi_n = (\psi_0^{\text{Inv}}(\nu_n)(\cdot) - \mu)$ , then (19) gives with  $t = 1$

$$\|\psi_0^{\text{Inv}}(\nu_n)(\cdot) - \mu\| \leq \|\nu_n\| \|c_{\varphi_n}\|^{-1} \leq \|\nu_n\| (\min\{\tau_l, 1 - \tau_u\})^{-1} \rightarrow 0.$$

Then (20) follows by continuity of  $\mu$ , see Lemma 13.  $\square$

*Proof of Lemma 9.* Let  $t_n \rightarrow 0$ ,  $t_n > 0$ ,  $(\varphi_n)_n \subset \ell^\infty[\tau_l, \tau_u]$  with  $\varphi_n \rightarrow \varphi \in \mathcal{C}[\tau_l, \tau_u]$  with respect to  $d_{\text{hyppi}}$  and thus uniformly by Proposition 2.1 in Bücher et al. (2014). From (19), and using the notation (22) we can write

$$t_n^{-1}(\psi_0^{\text{Inv}}(t_n \varphi_n) - \psi_0^{\text{Inv}}(0)) = \varphi_n / c_{\kappa_n}, \quad \kappa_n(\tau) = \psi_0^{\text{Inv}}(t_n \varphi_n)(\tau) - \mu_\tau$$

and we need to show that

$$\varphi_n/c_{\kappa_n} \rightarrow \psi^{\text{Inv}}(\varphi) = \varphi/c_0 \quad (23)$$

with respect to  $d_{\text{hyppi}}$ , where  $c_0$  is as in (22) with  $\varphi = 0$ .

Now, since  $\varphi_n \rightarrow \varphi$  uniformly and  $\varphi$  is continuous, to obtain (23) it suffices by Lemma 19, i) and iii), to show that  $c_{\kappa_n} \rightarrow c_0$  under  $d_{\text{hyppi}}$ . To this end, by Lemma A.4, Bücher et al. (2014) and Lemma 19, iii), it suffices to show that under  $d_{\text{hyppi}}$

$$h_n(\tau) = \int_0^1 F(\mu_\tau + s(\psi_0^{\text{Inv}}(t_n \varphi_n)(\tau) - \mu_\tau)) \, ds \rightarrow F(\mu_\tau) = h(\tau), \quad (24)$$

for which we shall use Corollary A.7 in Bücher et al. (2014). Let

$$\mathbb{T} = [\tau_l, \tau_u], \quad \mathcal{S} = \mathbb{T} \setminus \{\tau \in [\tau_l, \tau_u] \mid F \text{ is not continuous in } \mu_\tau\},$$

so that  $\mathcal{S}$  is dense in  $\mathbb{T}$  and  $h|_{\mathcal{S}}$  is continuous. Using the notation from Bücher et al. (2014), Appendix A.2, we have that

$$(h|_{\mathcal{S}})^{\mathcal{S}:\mathbb{T}}_{\wedge} = h_{\wedge} = F(\mu_{\cdot-}) \quad \text{and} \quad (h|_{\mathcal{S}})^{\mathcal{S}:\mathbb{T}}_{\vee} = h_{\vee} = h, \quad (25)$$

where the first equalities follow from the discussion in Bücher et al. (2014), Appendix A.2, and the second equalities from Lemma 19, ii) below, and where  $F(x-) = \lim_{t \uparrow x} F(t)$  denotes the left-sided limit of  $F$  at  $x$ . If we show that

- (i) for all  $\tau \in [\tau_l, \tau_u]$  with  $\tau_n \rightarrow \tau$  it holds that  $\liminf_n h_n(\tau_n) \geq F(\mu_{\tau-})$  and
- (ii) for all  $\tau \in [\tau_l, \tau_u]$  with  $\tau_n \rightarrow \tau$  it holds that  $\limsup_n h_n(\tau_n) \leq F(\mu_\tau)$ ,

Corollary A.7 in Bücher et al. (2014) implies (24), which concludes the proof of the convergence in (23).

To this end, concerning (i), we compute that

$$\begin{aligned} F(\mu_{\tau-}) &\leq \int_0^1 \liminf_n F(\mu_{\tau_n} + s(\psi_0^{\text{Inv}}(t_n \varphi_n)(\tau_n) - \mu_{\tau_n})) \, ds \\ &\leq \liminf_n \int_0^1 F(\mu_{\tau_n} + s(\psi_0^{\text{Inv}}(t_n \varphi_n)(\tau_n) - \mu_{\tau_n})) \, ds = \liminf_n h_n(\tau_n), \end{aligned}$$

where the first inequality follows from (20) and the fact that  $F(\mu_{\tau-}) \leq F(\mu_\tau)$ , and the second inequality follows from Fatou's lemma. For (ii) we argue analogously

$$\begin{aligned} F(\mu_\tau) &\geq \int_0^1 \limsup_n F(\mu_{\tau_n} + s(\psi_0^{\text{Inv}}(t_n \varphi_n)(\tau_n) - \mu_{\tau_n})) \, ds \\ &\geq \limsup_n \int_0^1 F(\mu_{\tau_n} + s(\psi_0^{\text{Inv}}(t_n \varphi_n)(\tau_n) - \mu_{\tau_n})) \, ds = \limsup_n h_n(\tau_n). \end{aligned}$$

This concludes the proof of the lemma. □

## 7.2 Details for the proof of Theorem 4

We let  $\psi_n^*(\varphi)(\tau) = -I_\tau(\varphi(\tau), F_n^*)$ ,  $\varphi \in \ell^\infty[\tau_l, \tau_u]$ , and denote by  $P_n^*$  the conditional law of  $Y_1^*, \dots, Y_n^*$  given  $Y_1, \dots, Y_n$ , and by  $\mathbb{E}_n^*$  expectation under this conditional law.

**Lemma 15.** *We have, almost surely, conditionally on  $Y_1, Y_2, \dots$ , the following statements.*

(i) *If  $\mathbb{E}[|Y|] < \infty$ , then*

$$\sup_{\tau \in [\tau_l, \tau_u]} |\mu_{\tau, n}^* - \hat{\mu}_{\tau, n}| = o_{P_n^*}(1). \quad (26)$$

Now assume  $\mathbb{E}[Y^2] < \infty$ .

(ii) *Weakly in  $(\ell^\infty[\tau_l, \tau_u], \|\cdot\|)$  it holds that*

$$\sqrt{n}(\psi_n^*(\hat{\mu}_{\cdot, n}) - \psi_n(\hat{\mu}_{\cdot, n})) \rightarrow Z \quad (27)$$

with  $Z$  as in Theorem 1.

(iii) *For every sequence  $\delta_n \rightarrow 0$  it holds that*

$$\begin{aligned} \sup_{\|\varphi\| \leq \delta_n} \sup_{\tau \in [\tau_l, \tau_u]} \sqrt{n} |\psi_n^*(\hat{\mu}_{\cdot, n} + \varphi)(\tau) - \psi_n(\hat{\mu}_{\cdot, n} + \varphi)(\tau) \\ - [\psi_n^*(\hat{\mu}_{\cdot, n})(\tau) - \psi_n(\hat{\mu}_{\cdot, n})(\tau)]| = o_{P_n^*}(1). \end{aligned} \quad (28)$$

(iv) *Weakly in  $(\ell^\infty[\tau_l, \tau_u], \|\cdot\|)$  we have that*

$$\sqrt{n}(\psi_n(\mu_{\cdot, n}^*) - \psi_n(\hat{\mu}_{\cdot, n})) \rightarrow Z. \quad (29)$$

*Proof of Lemma 15.* First consider (26). We start with individual consistency, the proof of which is inspired by Lemma 5.10, van der Vaart (2000). Since  $I_\tau(\mu_{\tau, n}^*, F_n^*) = 0$  and  $x \mapsto I_\tau(x, F_n^*)$  is strictly decreasing, for any  $\varepsilon, \eta_l, \eta_u > 0$  the inequality  $I_\tau(\hat{\mu}_{\tau, n} - \varepsilon, F_n^*) > \eta_l$  implies  $\mu_{\tau, n}^* > \hat{\mu}_{\tau, n} - \varepsilon$  and from  $I_\tau(\hat{\mu}_{\tau, n} + \varepsilon, F_n^*) < -\eta_u$  it follows that  $\mu_{\tau, n}^* < \hat{\mu}_{\tau, n} + \varepsilon$ . Thus

$$P_n^*(I_\tau(\hat{\mu}_{\tau, n} - \varepsilon, F_n^*) > \eta_l, I_\tau(\hat{\mu}_{\tau, n} + \varepsilon, F_n^*) < -\eta_u) \leq P_n^*(\hat{\mu}_{\tau, n} - \varepsilon < \mu_{\tau, n}^* < \hat{\mu}_{\tau, n} + \varepsilon)$$

and it suffices to show almost sure convergence of the left hand side to 1 for appropriately chosen  $\eta_l, \eta_u > 0$ , for which it is enough to deduce  $P_n^*(I_\tau(\hat{\mu}_{\tau, n} - \varepsilon, F_n^*) > \eta_l) \rightarrow 1$  and  $P_n^*(I_\tau(\hat{\mu}_{\tau, n} + \varepsilon, F_n^*) < -\eta_u) \rightarrow 1$  almost surely. Choose  $2\eta_l = I_\tau(\mu_\tau - \varepsilon, F) \neq 0$  to obtain

$$\begin{aligned} P_n^*(I_\tau(\hat{\mu}_{\tau, n} - \varepsilon, F_n^*) > \eta_l) &\geq P_n^*(|I_\tau(\mu_\tau - \varepsilon, F) - I_\tau(\hat{\mu}_{\tau, n} - \varepsilon, F_n^*) - I_\tau(\mu_\tau - \varepsilon, F)| > \eta_l) \\ &\geq P_n^*(|I_\tau(\hat{\mu}_{\tau, n} - \varepsilon, F_n^*) - I_\tau(\mu_\tau - \varepsilon, F)| < \eta_l) \end{aligned}$$

by the inverse triangle inequality. Similar for  $2\eta_u = I_\tau(\mu_\tau + \varepsilon, F) \neq 0$  we get that

$$P_n^*(I_\tau(\mu_{\tau, n}^* + \varepsilon, F_n^*) < -\eta_u) \geq P_n^*(|I_\tau(\mu_{\tau, n}^* + \varepsilon, F_n^*) - I_\tau(\mu_\tau + \varepsilon, F)| < \eta_u).$$

In both inequalities the right hand side converges to 1 almost surely, provided that almost surely,  $I_\tau(\hat{\mu}_{\tau, n} \pm \varepsilon, F_n^*) \rightarrow I_\tau(\mu_\tau \pm \varepsilon, F)$  in probability conditionally on  $Y_1, Y_2, \dots$ . To this end, start with

$$\mathbb{E}_n^*[I_\tau(\hat{\mu}_{\tau, n} \pm \varepsilon, F_n^*)] = \sum_{i=1}^n P_n^*(Y_1^* = Y_i) I_\tau(\hat{\mu}_{\tau, n} \pm \varepsilon, Y_i) = I_\tau(\hat{\mu}_{\tau, n} \pm \varepsilon, F_n),$$

so that it remains to show convergence of the right hand side to  $I_\tau(\mu_\tau \pm \varepsilon, F)$  for almost every sequence  $Y_1, Y_2, \dots$ . To this end it holds that

$$|I_\tau(\hat{\mu}_{\tau,n} \pm \varepsilon, F_n) - I_\tau(\mu_\tau \pm \varepsilon, F_n)| \leq |\hat{\mu}_{\tau,n} - \mu_\tau|$$

by Lemma 13, where the bound converges to 0 by the strong consistency of  $\hat{\mu}_{\tau,n}$  (Holzmann and Klar, 2016, Theorem 2). Further  $I_\tau(\mu_\tau \pm \varepsilon, F_n) \rightarrow I_\tau(\mu_\tau \pm \varepsilon, F)$  almost surely by the strong law of large numbers, thus  $I_\tau(\hat{\mu}_{\tau,n} \pm \varepsilon, F_n)$  converges to  $I_\tau(\mu_\tau \pm \varepsilon, F)$  for almost every sequence  $Y_1, Y_2, \dots$ , what concludes the proof of individual consistency of  $\mu_{\tau,n}^*$ . To strengthen this to uniform consistency, we use a Glivenko-Cantelli argument as in Holzmann and Klar (2016), Theorem 2. Let  $d_n = \hat{\mu}_{\tau_u,n} - \hat{\mu}_{\tau_l,n}$  and observe that  $d_n \rightarrow d = \mu_{\tau_u} - \mu_{\tau_l}$  almost surely by consistency of  $\hat{\mu}_{\tau,n}$ . Let  $m \in \mathbb{N}$  and choose  $\tau_l = \tau_0 \leq \tau_1 \leq \dots \leq \tau_m = \tau_u$  such that

$$\hat{\mu}_{\tau_k,n} = \hat{\mu}_{\tau_l,n} + \frac{k d_n}{m} \quad (k \in \{1, \dots, m\}),$$

which is possible because of the continuity of  $\tau \mapsto \hat{\mu}_{\tau,n}$ . Because the expectile functional is strictly increasing in  $\tau$  it follows that

$$\mu_{\tau_k,n}^* - \hat{\mu}_{\tau_{k+1},n} \leq \mu_{\tau,n}^* - \hat{\mu}_{\tau,n} \leq \mu_{\tau_{k+1},n}^* - \hat{\mu}_{\tau_k,n} \quad (\tau_k \leq \tau \leq \tau_{k+1}).$$

This implies

$$\sup_{\tau \in [\tau_l, \tau_u]} |\mu_{\tau,n}^* - \hat{\mu}_{\tau,n}| \leq \max_{1 \leq k \leq m} |\mu_{\tau_k,n}^* - \hat{\mu}_{\tau_k,n}| + \frac{d_n}{m},$$

hence

$$\limsup_n \|\mu_{\tau,n}^* - \hat{\mu}_{\tau,n}\|_{[\tau_l, \tau_u]} \leq \limsup_n \max_{1 \leq k \leq m} |\mu_{\tau_k,n}^* - \hat{\mu}_{\tau_k,n}| + \limsup_n \frac{d_n}{m} = \frac{d}{m}$$

conditionally in probability for almost every sequence  $Y_1, Y_2, \dots$ . Letting  $m \rightarrow \infty$  completes the proof.

*Proof of (27).* The idea is to use Theorem 19.28, van der Vaart (2000), for the random class

$$\mathcal{F}_n = \{y \mapsto -I_\tau(\hat{\mu}_{\tau,n}, y) \mid \tau \in [\tau_l, \tau_u]\},$$

which is a subset of

$$\mathcal{F}_{\eta_n} = \{y \mapsto -I_\tau(\mu_\tau + x, y) \mid |x| \leq \eta_n, \tau \in [\tau_l, \tau_u]\}$$

for the sequence  $\eta_n = \sup_{\tau \in [\tau_l, \tau_u]} |\hat{\mu}_{\tau,n} - \mu_\tau|$ , hence, almost surely,

$$J_{[]}(\varepsilon, \mathcal{F}_n, L_2(F_n)) \leq J_{[]}(\varepsilon, \mathcal{F}_{\eta_n}, L_2(F_n)).$$

The class  $\mathcal{F}_{\eta_n}$  has envelope  $(|\mu_{\tau_l}| + |\mu_{\tau_u}| + \eta_n + y)$ , which satisfies the Lindeberg condition.

By (17) the class  $\mathcal{F}_{\eta_n}$  is a class consisting of Lipschitz-functions with Lipschitz-constant  $m_n(y) = (C + \eta_n + |y|)$  for some  $C \geq 1$ , such that the bracketing number fulfils

$$N_{[]}(\delta, \mathcal{F}_{\eta_n}, L_2(F_n)) \leq K \left\{ \mathbb{E}_n [m_n(Y)^2] \frac{\eta_n + \tau_u - \tau_l}{\delta} \right\}^2$$

by Example 19.7, van der Vaart (2000), where  $K$  is some constant not depending on  $n$ . By the strong law of large numbers,  $\mathbb{E}_n [m_n(Y)^2] \rightarrow \mathbb{E} [(C + |Y|)^2]$  holds almost surely, in addition



$\eta_n \rightarrow 0$  almost surely by the strong consistency of  $\hat{\mu}_{\tau,n}$ , such that the above bracketing number is of order  $1/\delta^2$ . Thus the bracketing integral  $J_{[]}(\varepsilon_n, \mathcal{F}_{\eta_n}, L_2(F_n))$  converges to 0 almost surely for every sequence  $\varepsilon_n \searrow 0$ .

Next we show the almost sure convergence of the expectation  $\mathbb{E}_n [I_\tau(\hat{\mu}_{\tau,n}, Y) I_{\tau'}(\hat{\mu}_{\tau',n}, Y)]$  to  $\mathbb{E} [I_\tau(\mu_\tau, Y) I_{\tau'}(\mu_{\tau'}, Y)]$ . First it holds that

$$\begin{aligned} \mathbb{E}_n [I_\tau(\hat{\mu}_{\tau,n}, Y) I_{\tau'}(\hat{\mu}_{\tau',n}, Y)] &= \mathbb{E}_n [I_\tau(\hat{\mu}_{\tau,n}, Y) (I_{\tau'}(\hat{\mu}_{\tau',n}, Y) - I_{\tau'}(\mu_{\tau'}, Y))] \\ &\quad + \mathbb{E}_n [I_{\tau'}(\mu_{\tau'}, Y) (I_\tau(\hat{\mu}_{\tau,n}, Y) - I_\tau(\mu_\tau, Y))] \\ &\quad + \mathbb{E}_n [I_\tau(\mu_\tau, Y) I_{\tau'}(\mu_{\tau'}, Y)], \end{aligned}$$

where the last summand converges almost surely to  $\mathbb{E} [I_\tau(\mu_\tau, Y) I_{\tau'}(\mu_{\tau'}, Y)]$  by the strong law of large numbers. For the first term we estimate

$$\mathbb{E}_n [|I_\tau(\hat{\mu}_{\tau,n}, Y)| |I_{\tau'}(\hat{\mu}_{\tau',n}, Y) - I_{\tau'}(\mu_{\tau'}, Y)|] \leq \{|\hat{\mu}_{\tau,n}| + \mathbb{E}_n [|Y|]\} |\hat{\mu}_{\tau',n} - \mu_{\tau'}|$$

with aid of Lemma 13, where the upper bound converges to 0 almost surely by the strong law of large numbers, strong consistency of  $\hat{\mu}_{\tau',n}$  and boundedness of  $\hat{\mu}_{\tau,n}$ , which in fact also follows from the strong consistency of the empirical expectile and since  $\tau \in [\tau_l, \tau_u]$ . The remaining summand above is treated likewise, hence  $\mathbb{E}_n [I_\tau(\hat{\mu}_{\tau,n}, Y) I_{\tau'}(\hat{\mu}_{\tau',n}, Y)]$  indeed converges almost surely to the asserted limit. The assertion now follows from Theorem 19.28, van der Vaart (2000).

*Proof of (28).* Set  $\eta_n = \sup_{\tau \in [\tau_l, \tau_u]} |\hat{\mu}_{\tau,n} - \mu_\tau|$  again, then as a first step we can estimate

$$\begin{aligned} &\sup_{\|\varphi\| \leq \delta_n} \sup_{\tau \in [\tau_l, \tau_u]} \sqrt{n} |\psi_n^*(\hat{\mu}_{\cdot,n} + \varphi)(\tau) - \psi_n(\hat{\mu}_{\cdot,n} + \varphi)(\tau) - [\psi_n^*(\hat{\mu}_{\cdot,n})(\tau) - \psi_n(\hat{\mu}_{\cdot,n})(\tau)]| \\ &= \sup_{\|\varphi\| \leq \delta_n} \sup_{\tau \in [\tau_l, \tau_u]} \sqrt{n} |\psi_n^*\{\mu_{\cdot} + (\hat{\mu}_{\cdot,n} - \mu_{\cdot} + \varphi)\}(\tau) - \psi_n\{\mu_{\cdot} + (\hat{\mu}_{\cdot,n} - \mu_{\cdot} + \varphi)\}(\tau) \\ &\quad - [\psi_n^*\{\mu_{\cdot} + (\hat{\mu}_{\cdot,n} - \mu_{\cdot})\}(\tau) - \psi_n\{\mu_{\cdot} + (\hat{\mu}_{\cdot,n} - \mu_{\cdot})\}(\tau)]| \\ &\leq \sup_{|x_1|, |x_2| \leq \delta_n + \eta_n} \sup_{\tau \in [\tau_l, \tau_u]} \sqrt{n} |\psi_n^*(\mu_{\cdot} + x_1)(\tau) - \psi_n(\mu_{\cdot} + x_1)(\tau) \\ &\quad - [\psi_n^*(\mu_{\cdot} + x_2)(\tau) - \psi_n(\mu_{\cdot} + x_2)(\tau)]|, \end{aligned}$$

so that for (28) it suffices to show the almost sure convergence  $\mathbb{E}_n^* [\|\sqrt{n}(F_n^* - F_n)\|_{\mathcal{F}_{\rho_n}}] \rightarrow 0$  for the class

$$\mathcal{F}_{\rho_n} = \{y \mapsto I_\tau(\mu_\tau + x_1, y) - I_\tau(\mu_\tau + x_2, y) \mid |x_1|, |x_2| \leq \rho_n, \tau \in [\tau_l, \tau_u]\}, \quad \rho_n = \delta_n + \eta_n.$$

We use Corollary 19.35, van der Vaart (2000), which implies that

$$\mathbb{E}_n^* [\|\sqrt{n}(F_n^* - F_n)\|_{\mathcal{F}_{\rho_n}}] \leq J_{[]}[\mathbb{E}_n [m_n(Y)^2], \mathcal{F}_{\rho_n}, L_2(F_n)]$$

almost surely, where  $m_n(y)$  is an envelope function for  $\mathcal{F}_{\rho_n}$ . The proof consists of finding this envelope and determining the order of the bracketing integral. By using Lemma 13 the class  $\mathcal{F}_{\rho_n}$  consists of Lipschitz-functions, as for any  $\tau, \tau' \in [\tau_l, \tau_u]$ ,  $x_1, x'_1, x_2, x'_2 \in [-\rho_n, \rho_n]$  the almost sure inequality

$$\begin{aligned} &|I_\tau(\mu_\tau + x_1, y) - I_\tau(\mu_\tau + x_2, y) - (I_{\tau'}(\mu_{\tau'} + x'_1, y) - I_{\tau'}(\mu_{\tau'} + x'_2, y))| \\ &\leq (|x_1 - x'_1| + |x_2 - x'_2| + |\tau - \tau'|) 2(C + \rho_n + |y|) \end{aligned} \quad (30)$$

is true for some constant  $C > 0$ , see also (17). Thus the bracketing integral  $J_{[]}(\varepsilon_n, \mathcal{F}_{\rho_n}, L_2(F_n))$  converges to 0 almost surely for any sequence  $\varepsilon_n \rightarrow 0$  with the same arguments as above. Finally, by (30) the function  $m_n(y) = \eta_n 4(C + |y|)$  is an envelope for  $\mathcal{F}_{\rho_n}$ . By the strong law of large numbers, the square integrability of  $Y$  and since  $\eta_n \rightarrow 0$  almost surely it holds that  $\mathbb{E}_n [m_n(Y)^2] \rightarrow 0$  almost surely, so that

$$\mathbb{E}_n^* [\|\sqrt{n}(F_n^* - F_n)\|_{\mathcal{F}_{\rho_n}}] \leq J_{[]}[\mathbb{E}_n [m_n(Y)^2], \mathcal{F}_{\rho_n}, L_2(F_n)] \rightarrow 0$$

is valid for almost every sequence  $Y_1, Y_2, \dots$ , which concludes the proof.

*Proof of (29).* By (26) and (28), and since  $\psi_n(\hat{\mu}_{\cdot, n}) = \psi_n^*(\mu_{\cdot, n}^*) = 0$ , we have that almost surely,

$$\begin{aligned} \sqrt{n}(\psi_n(\mu_{\cdot, n}^*) - \psi_n(\hat{\mu}_{\cdot, n})) &= \sqrt{n}(\psi_n(\mu_{\cdot, n}^*) - \psi_n^*(\mu_{\cdot, n}^*)) \\ &= -\sqrt{n}(\psi_n^*(\hat{\mu}_{\cdot, n}) - \psi_n(\hat{\mu}_{\cdot, n})) + o_{\mathbb{P}_n^*}(1) \end{aligned}$$

The right hand side converges to  $-Z$  conditionally in distribution, almost surely, by (27), which equals  $Z$  in distribution.  $\square$

**Lemma 16.** *The map  $\psi_n$  is invertible. Further, if  $t_n \rightarrow 0$ ,  $\varphi \in \mathcal{C}[\tau_l, \tau_u]$  and  $\varphi_n \rightarrow \varphi$  with respect to  $d_{\text{hyppi}}$  and hence uniformly, we have that almost surely, conditionally on  $Y_1, Y_2, \dots$ ,*

$$t_n^{-1}(\psi_n^{\text{Inv}}(t_n \varphi_n) - \psi_n^{\text{Inv}}(0)) \rightarrow \dot{\psi}^{\text{Inv}}(\varphi) \quad (31)$$

with respect to the hypi-semimetric.

*Proof.* The first part follows from Lemma 8 with  $F$  in  $\psi_0$  replaced by  $F_n$  in  $\psi_n$  as no specific assumptions on  $F$  were used in that lemma. For (31), with the same calculations as for Lemma 14 we obtain the representation

$$\begin{aligned} &t_n^{-1}(\psi_n^{\text{Inv}}(t_n \varphi_n) - \psi_n^{\text{Inv}}(0))(\tau) \\ &= \varphi_n(\tau) \left\{ \tau + (1 - 2\tau) \int_0^1 F_n(\hat{\mu}_{\tau, n} + s(\psi_n^{\text{Inv}}(t_n \varphi_n)(\tau) - \hat{\mu}_{\tau, n})) \, ds \right\}^{\text{Inv}}, \end{aligned}$$

and we have to prove hypi convergence to  $\dot{\psi}^{\text{Inv}}(\varphi)$ . By the same reductions as in the proof of Theorem 9, it suffices to prove the hypi-convergence of

$$h_n(\tau) = \int_0^1 F_n(\hat{\mu}_{\tau, n} + s(\psi_n^{\text{Inv}}(t_n \varphi_n)(\tau) - \hat{\mu}_{\tau, n})) \, ds$$

to  $F(\mu_\tau)$  for almost every sequence  $Y_1, Y_2, \dots$ . To this end, observe that for any  $s \in [0, 1]$  the sequence  $\hat{\mu}_{\tau, n} + s(\psi_n^{\text{Inv}}(t_n \varphi_n)(\tau) - \hat{\mu}_{\tau, n})$  converges to  $\mu_\tau$  almost surely by the same arguments as in Lemma 14. Since  $\mu_\tau$  is continuous in  $\tau$ , the almost sure convergence  $\hat{\mu}_{\tau_n, n} + s(\psi_n^{\text{Inv}}(t_n \varphi_n)(\tau_n) - \hat{\mu}_{\tau_n, n}) \rightarrow \mu_\tau$  holds for any sequence  $\tau_n \rightarrow \tau$ . By adding and subtracting

$F_n(\hat{\mu}_{\tau,n} + s(\psi_n^{\text{Inv}}(t_n \varphi_n)(\tau) - \hat{\mu}_{\tau,n}))$  and using Lemma 17, we now can estimate

$$\begin{aligned} F(\mu_{\tau-}) &\leq \int_0^1 \liminf_n \left( F_n(\hat{\mu}_{\tau,n} + s(\psi_n^{\text{Inv}}(t_n \varphi_n)(\tau) - \hat{\mu}_{\tau,n})) \right) \\ &+ \limsup_n \left( F(\hat{\mu}_{\tau,n} + s(\psi_n^{\text{Inv}}(t_n \varphi_n)(\tau) - \hat{\mu}_{\tau,n})) - F_n(\hat{\mu}_{\tau,n} + s(\psi_n^{\text{Inv}}(t_n \varphi_n)(\tau) - \hat{\mu}_{\tau,n})) \right) ds \\ &\leq \int_0^1 \liminf_n F_n(\hat{\mu}_{\tau,n} + s(\psi_n^{\text{Inv}}(t_n \varphi_n)(\tau) - \hat{\mu}_{\tau,n})) ds + \limsup_n \|F_n - F\|_{\mathbb{R}} \\ &\leq \liminf_n \int_0^1 F_n(\hat{\mu}_{\tau,n} + s(\psi_n^{\text{Inv}}(t_n \varphi_n)(\tau) - \hat{\mu}_{\tau,n})) ds = \liminf_n h_n(\tau_n) \end{aligned}$$

almost surely, where the 'lim sup' vanishes due to the Glivenko-Cantelli-Theorem for the empirical distribution function. Similarly we almost surely have

$$\begin{aligned} F(\mu_{\tau}) &\geq \int_0^1 \limsup_n F_n(\hat{\mu}_{\tau,n} + s(\psi_n^{\text{Inv}}(t_n \varphi_n)(\tau) - \hat{\mu}_{\tau,n})) ds + \liminf_n \|F_n - F\|_{\mathbb{R}} \\ &\geq \limsup_n \int_0^1 F_n(\hat{\mu}_{\tau,n} + s(\psi_n^{\text{Inv}}(t_n \varphi_n)(\tau) - \hat{\mu}_{\tau,n})) ds = \limsup_n h_n(\tau_n). \end{aligned}$$

The proof is concluded as that of Theorem 9 by using Corollary A.7, Bücher et al. (2014).  $\square$

*Proof of Theorem 4.* Set  $t_n = 1/\sqrt{n}$  and define the function  $g_n(\varphi) = t_n^{-1}(\psi_n^{\text{Inv}}(t_n \varphi) - \psi_n^{\text{Inv}}(0))$ . Then from (31) the hypi-convergence  $g_n(\varphi_n) \rightarrow \psi^{\text{Inv}}(\varphi)$  holds almost surely, whenever  $\varphi \in \mathcal{C}[\tau_l, \tau_u]$  and  $\varphi_n \rightarrow \varphi$  with respect to  $d_{\text{hypi}}$ . In addition  $\sqrt{n}(\psi_n(\mu_{\cdot,n}^*) - \psi_n(\hat{\mu}_{\cdot,n})) \rightarrow Z$  conditional in distribution with respect to the sup-norm, almost surely, by (27), where  $Z$  is continuous almost surely. Hence the convergence is also valid with respect to  $d_{\text{hypi}}$ , such that

$$\sqrt{n}(\mu_{\cdot,n}^* - \hat{\mu}_{\cdot,n}) = g_n(\sqrt{n}(\psi_n(\mu_{\cdot,n}^*) - \psi_n(\hat{\mu}_{\cdot,n}))) \rightarrow \psi^{\text{Inv}}(Z)$$

holds conditionally in distribution, almost surely, by using the extended continuous mapping theorem, Theorem B.3, in Bücher et al. (2014).  $\square$

### 7.3 Details for the proof of Theorem 5

Since  $F$  in Theorem 5 is assumed to be continuous and strictly increasing, it is differentiable almost everywhere. If  $F$  is not differentiable in a point  $x$ , the assumptions at least guarantee right- and left-sided limits of  $f$  in  $x$ . Since redefining a density on a set of measure zero is possible without changing the density property, we can assume  $f(x) = f(x-)$  for such  $x$ . In addition, the assumptions imply that  $F$  is invertible with continuous inverse, which we denote by  $F^{\text{Inv}}$ .

*Proof of Lemma 10.* Let  $t_n \searrow 0$ ,  $\alpha_n \rightarrow \alpha \in [\alpha_l, \alpha_u]$  and  $\varphi_n, \nu_n \in \ell^\infty[\alpha_l, \alpha_u]$  with  $\varphi_n \rightarrow \varphi \in \mathcal{C}[\alpha_l, \alpha_u]$  and  $\nu_n \rightarrow 0$  with respect to  $d_{\text{hypi}}$ . Then  $\varphi_n(\alpha_n) \rightarrow \varphi(\alpha)$  and  $\nu_n(\alpha_n) \rightarrow 0$  by Proposition 2.1, Bücher et al. (2014). We have to deal with the limes inferior and superior of

$$t_n^{-1}(F^{\text{Inv}}(\alpha_n + \nu_n(\alpha_n) + t_n \varphi_n(\alpha_n)) - F^{\text{Inv}}(\alpha_n + \nu_n(\alpha_n))).$$

Define the function  $g : [0, 1] \rightarrow \mathbb{R}$  with  $g(s) = F^{\text{Inv}}(\alpha_n + \nu_n(\alpha_n) + s t_n \varphi_n(\alpha_n))$ . The difference above then equals  $t_n^{-1}(g(1) - g(0))$ . The function  $g$  is monotonically increasing (or decreasing,

depending on the sign of  $\varphi_n(\alpha_n)$  and continuous in  $s$ . Hence with Theorem 7.23, Thomson et al. (2008), we can write

$$\begin{aligned} t_n^{-1}(g(1) - g(0)) &= \int_0^1 g'(s) ds + \mu_g(\{g' = \pm\infty\}) \\ &= \varphi_n(\alpha_n) \int_0^1 \left[ f\{F^{\text{Inv}}(\alpha_n + \nu_n(\alpha_n) + s t_n \varphi_n(\alpha_n))\} \right]^{-1} ds, \end{aligned} \quad (32)$$

where  $\mu_g$  is the Lebesgue-Stieltjes signed measure associated with  $g$ . The set  $\{g' = \pm\infty\}$  is empty, as  $g' = \infty$  if and only if  $f = 0$ , which is excluded by assumption, so that  $\lambda_g(\{g' = \pm\infty\}) = 0$ . Since  $\varphi_n \rightarrow \varphi \in \mathcal{C}[\alpha_l, \alpha_u]$ , using Lemma 19 i) we only need to deal with the accumulation points of the integral in (32). To this end let

$$H_n(\alpha) = \int_0^1 \left( f\{F^{\text{Inv}}(\alpha_n + \nu_n(\alpha_n) + s t_n \varphi_n(\alpha_n))\} \right)^{-1} ds.$$

The assertion is that  $H_n(\alpha)$  hypi-converges to  $f(q_\alpha)^{-1}$ . In the case for the expectile-process we were able to exploit continuity properties of the asserted semi-derivative and thus utilize Corollary A.7, Bücher et al. (2014)). In the present context we do not know whether the limit  $f(q_\alpha)^{-1}$  can be attained by extending  $f(q_\alpha)^{-1}$  from the dense set where it is continuous. Thus, we directly show the defining properties of hypi convergence in (6) and (7). For any  $s \in [0, 1]$  it holds that  $F^{\text{Inv}}(\alpha_n + \nu_n(\alpha_n) + s t_n \varphi_n(\alpha_n)) \rightarrow q_\alpha$  by continuity of  $F^{\text{Inv}}$ , hence we can estimate

$$\begin{aligned} \liminf_n \int_0^1 \left( f\{F^{\text{Inv}}(\alpha_n + \nu_n(\alpha_n) + s t_n \varphi_n(\alpha_n))\} \right)^{-1} ds \\ \geq \int_0^1 \left( \limsup_n f\{F^{\text{Inv}}(\alpha_n + \nu_n(\alpha_n) + s t_n \varphi_n(\alpha_n))\} \right)^{-1} ds \geq \frac{1}{f_\vee(q_\alpha)}. \end{aligned}$$

with help of Fatou's Lemma, Lemma 17 and the properties of the hulls. Similarly,

$$\begin{aligned} \limsup_n \int_0^1 \left( f\{F^{\text{Inv}}(\alpha_n + \nu_n(\alpha_n) + s t_n \varphi_n(\alpha_n))\} \right)^{-1} ds \\ \leq \int_0^1 \left( \liminf_n f\{F^{\text{Inv}}(\alpha_n + \nu_n(\alpha_n) + s t_n \varphi_n(\alpha_n))\} \right)^{-1} ds \leq \frac{1}{f_\wedge(q_\alpha)}. \end{aligned}$$

It remains to construct sequences  $\alpha_n$  such that  $\lim_n H_n(\alpha_n)$  equals the respective hull. By Lemma 19 ii), it holds that

$$f_\wedge(q_\alpha) = \min \{f(q_{\alpha-}), f(q_{\alpha+}), f(q_\alpha)\} \quad \text{and} \quad f(q_\alpha) = \max \{f(q_{\alpha-}), f(q_{\alpha+}), f(q_\alpha)\}.$$

If  $F$  is differentiable in  $q_\alpha$ , all three values can differ; if not, we have set  $f(q_\alpha) = f(q_{\alpha-})$ . We shall choose a sequence converging from below, above or which is always equal to  $\alpha$ , depending on where the min / max is attained. We do this exemplary for  $f_\vee(q_\alpha) = f(q_{\alpha-})$ . Since  $\varphi_n \rightarrow \varphi$  uniformly and  $\varphi$  is bounded, there is a  $C > 0$  for which  $\|\varphi_n\| \leq C$ . Then choose any sequence  $\varepsilon_n \searrow 0$  and set  $\alpha_n = \alpha - \|\nu_n\| - t_n C - \varepsilon_n$ , which is in  $[\alpha_l, \alpha_u]$  for  $n$  big enough. For this sequence it holds that  $\alpha_n + \nu_n(\alpha_n) + s t_n \varphi_n(\alpha_n) < \alpha$  for any  $s \in [0, 1]$ , so that the convergence is from below. Bounded convergence yields

$$\begin{aligned} \lim_n \int_0^1 \left( f\{F^{\text{Inv}}(\alpha_n + \nu_n(\alpha_n) + s t_n \varphi_n(\alpha_n))\} \right)^{-1} ds \\ = \int_0^1 \left( \lim_n f\{F^{\text{Inv}}(\alpha_n + \nu_n(\alpha_n) + s t_n \varphi_n(\alpha_n))\} \right)^{-1} ds = \frac{1}{f(q_{\alpha-})}. \end{aligned}$$

□

## 7.4 Properties of hypi-convergence

An major technical issue in the above argument is to determine hypi-convergence of sums, products and quotients of hypi-convergent functions. We shall require the following basic relations between  $\limsup$  and  $\liminf$ .

**Lemma 17.** *Let  $(a_n)_n, (b_n)_n$  be bounded sequences. Then*

$$\liminf_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n \geq \liminf_{n \rightarrow \infty} (a_n + b_n) \geq \liminf_{n \rightarrow \infty} a_n + \liminf_{n \rightarrow \infty} b_n$$

and

$$\limsup_{n \rightarrow \infty} a_n + \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n.$$

If  $a_n \neq 0$ , then

$$\liminf_n \frac{1}{a_n} = \frac{1}{\limsup_n a_n}.$$

*Proof.* The first pair of inequalities follows from

$$a_k + \sup_{l \geq n} b_l \geq a_k + b_k \geq a_k + \inf_{l \geq n} b_l$$

for any  $n \in \mathbb{N}, k \geq n$ , by applying 'inf $_{k \geq n}$ ' and taking the limit  $n \rightarrow \infty$ . The second pair of inequalities follows similarly. For the last part note that

$$\inf_{l \geq n} \frac{1}{a_l} = \frac{1}{\sup_{l \geq n} a_l}.$$

Taking  $n \rightarrow \infty$  yields the asserted equality.  $\square$

**Lemma 18.** *Let  $(b_n)_n$  be a bounded sequence and let  $(a_n)_n$  be convergent with limit  $a \in \mathbb{R}$ . Then*

$$\liminf_n a_n b_n = \liminf_n a b_n, \quad \limsup_n a_n b_n = \limsup_n a b_n.$$

*Proof.* From Lemma 17,

$$\liminf_n (a_0 b_n - a_n b_n) + \liminf_n a_0 b_n \leq \liminf_n a_n b_n \leq \limsup_n (a_0 b_n - a_n b_n) + \liminf_n a_0 b_n,$$

$$\liminf_n (a_0 b_n - a_n b_n) + \limsup_n a_0 b_n \leq \limsup_n a_n b_n \leq \limsup_n (a_0 b_n - a_n b_n) + \limsup_n a_0 b_n.$$

But  $|a_0 b_n - a_n b_n| \leq \sup_m |b_m| |a_0 - a_n| \rightarrow 0$ , which implies the stated result.  $\square$

The next lemma was crucial in the proof of Theorem 9.

**Lemma 19.** *Let  $c, c_n, \varphi_n \in \ell^\infty([l, u])$  and  $\varphi \in \mathcal{C}[l, u]$ .*

i) *If  $d_{\text{hypi}}(\varphi_n, \varphi), d_{\text{hypi}}(c_n, c) \rightarrow 0$  holds, then  $c_n \varphi_n$  hypi-converges to  $c\varphi$ . More precisely  $c_n \varphi_n$  epi-converges to  $(c\varphi)_\wedge$  and hypo-converges to  $(c\varphi)_\vee$ , where*

$$(\varphi c)_\wedge = \varphi (c_\wedge 1_{\varphi > 0} + c_\vee 1_{\varphi < 0}), \quad (\varphi c)_\vee = \varphi (c_\vee 1_{\varphi > 0} + c_\wedge 1_{\varphi < 0}). \quad (33)$$

ii) Assume  $c$  has left- and right-sided limits at every point in  $[l, u]$ . Then

$$\begin{aligned} c_{\wedge}(x) &= \min\{c(x-), c(x), c(x+)\} & \text{and} \\ c_{\vee}(x) &= \max\{c(x-), c(x), c(x+)\} \end{aligned}$$

holds. Especially, if  $c(x_0-) \leq c(x_0) \leq c(x_0+)$  or  $c(x_0-) \geq c(x_0) \geq c(x_0+)$  for some  $x_0 \in [l, u]$ , then the equalities  $(c_{\wedge})_{\vee}(x_0) = c_{\vee}(x_0)$  and  $(c_{\vee})_{\wedge}(x_0) = c_{\wedge}(x_0)$  are true.

iii) If  $c_n, c > 0$ , the convergence  $d_{\text{hyppi}}(1/c_n, 1/c) \rightarrow 0$  follows from  $d_{\text{hyppi}}(c_n, c) \rightarrow 0$ .

*Proof of Lemma 19.* From the definition in (5), for a function  $h \in \ell^\infty[l, u]$  the lower semi-continuous hull  $h_{\wedge}$  at  $x \in [l, u]$  is characterized by the following conditions

$$\begin{aligned} \text{For any sequence } x_n \rightarrow x, \quad & \text{we have } \liminf_{n \rightarrow \infty} h(x_n) \geq h_{\wedge}(x), \\ \text{There is a sequence } x'_n \rightarrow x \quad & \text{for which } \lim_{n \rightarrow \infty} h(x'_n) = h_{\wedge}(x), \end{aligned} \tag{34}$$

and similarly for  $h_{\vee}$ .

Ad i): By continuity of  $\varphi$  we have  $\varphi(x_n) \rightarrow \varphi(x)$  for any sequence  $x_n \rightarrow x$ . The statement (33) now follows immediately using (34) and Lemma 18 and noting that for  $\varphi(x) < 0$ ,

$$\liminf_{n \rightarrow \infty} \varphi(x)c(x_n) = \varphi(x) \limsup_{n \rightarrow \infty} c(x_n), \quad \limsup_{n \rightarrow \infty} \varphi(x)c(x_n) = \varphi(x) \liminf_{n \rightarrow \infty} c(x_n).$$

Further, by continuity of  $\varphi$ , the hypi-convergence of  $\varphi_n$  to  $\varphi$  actually implies the uniform convergence. Therefore, for any  $x_n \rightarrow x$  we have that  $\varphi_n(x_n) \rightarrow \varphi(x)$ . Using the pointwise criteria (6) and (7) for hypi-convergence, Lemma 18 and (33) we obtain the asserted convergence  $\varphi_n c_n \rightarrow \varphi c$  with respect to the hypi semi-metric.

Ad ii): The proof of Lemma C.6, Bücher et al. (2014) show that for a function, which admits right- and left-sided limits, the supremum over a shrinking neighbourhood around a point  $x$  converges to the maximum of the three points  $c(x-)$ ,  $c(x)$  and  $c(x+)$ . The analogues statement holds for the infimum, which is the first part of ii). From Lemma C.5, Bücher et al. (2014), the maps  $x \mapsto c(x-)$  and  $x \mapsto c(x+)$  both have a right-sided limit equal to  $c(x+)$  and a left-sided limit equal to  $c(x-)$ , hence this is also true for the functions  $c_{\vee}(x) = \max\{c(x-), c(x), c(x+)\}$  and  $c_{\wedge}(x) = \min\{c(x-), c(x), c(x+)\}$ . From the above argument, we obtain

$$\begin{aligned} (c_{\vee})_{\wedge}(x) &= \min\{c(x-), c(x+), \max\{c(x-), c(x), c(x+)\}\} = \min\{c(x-), c(x+)\} & \text{and} \\ (c_{\wedge})_{\vee}(x) &= \max\{c(x-), c(x+), \min\{c(x-), c(x), c(x+)\}\} = \max\{c(x-), c(x+)\}. \end{aligned}$$

If  $c(x_0-) \leq c(x_0) \leq c(x_0+)$  or  $c(x_0-) \geq c(x_0) \geq c(x_0+)$ , we obtain

$$c_{\vee}(x_0) = \max\{c(x_0-), c(x_0+)\} = (c_{\wedge})_{\vee}(x_0), \quad c_{\wedge}(x_0) = \min\{c(x_0-), c(x_0+)\} = (c_{\vee})_{\wedge}(x_0).$$

Ad iii): From Lemma 17 (last statement) and (34) we obtain  $(1/c)_{\wedge} = 1/c_{\vee}$  and  $(1/c)_{\vee} = 1/c_{\wedge}$ . The hypi-convergence of  $1/c_n$  to these hulls follows similarly from Lemma 17 (last statement) and the pointwise criteria (6) and (7) for hypi-convergence.  $\square$