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Quartic surfaces with 16 skew conics

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0. Introduction

The aim of this note is to construct a two-dimensional family of smooth quartic surfaces in $\mathbb{P}_3$ containing 16 mutually disjoint conics and altogether exactly 432 conics.

For an integer $d \geq 1$ denote by $N_4(d)$ the maximal number of smooth rational curves of degree d that can lie on a smooth quartic surface in $\mathbb{P}_3$. Schur [9] gave an example of a smooth quartic surface with 64 lines and Segre [10] showed that there are no smooth quartics containing more than 64 lines, so

$$N_4(1) = 64 \ .$$

Up to now the numbers $N_4(d)$, $d \geq 2$, seem to be unknown. The result of [1] gives the lower bound

$$N_4(2) \geq 352 \ .$$

This was shown by embedding the smooth Kummer surface X of a *generic* polarized abelian surface $(A, \mathcal{L})$ of type $(1, 9)$ into $\mathbb{P}_3$ via a certain line bundle $\mathcal{M}_{\mathcal{L}}$ on X associated to $\mathcal{L}$. These quartic surfaces contain 16 disjoint conics, which according to Nikulin's theorem [7] is the maximal number of *disjoint* smooth rational curves that can lie on a smooth quartic surface.

In order to obtain an upper bound for $N_4(2)$ S.A. Strømme suggested to determine the number of conics in a generic pencil of quartic surfaces in $\mathbb{P}_3$. A Chern class computation by the methods of [5] shows that this number is 5016. Thus

$$N_4(2) \leq 5016 \ .$$

Although the number of conics in a pencil of quartics seems to be too rough an estimate for the number of conics on an individual quartic, this is the best upper bound available at present.

In the present paper we first classify the smooth quartic surfaces in $\mathbb{P}_3$ containing 16 skew conics in terms of their abelian covers, i.e. we determine the precise conditions on a polarized abelian surface $(A, \mathcal{L})$ of type $(1, 9)$ under which the associated line bundle $\mathcal{M}_{\mathcal{L}}$ on the smooth Kummer surface is very ample (see Sect. 2). Then, in Sect. 3, we consider principally polarized abelian surfaces with endomorphism ring

$\mathbb{Z}[\sqrt{7}]$. These surfaces carry a natural $(1,9)$-polarization. Using the classification given in Sect. 2 we show that their Kummer surfaces in fact embed into $\mathbb{P}_3$. The quartic surfaces constructed in this way contain exactly 432 conics. So we obtain the improved lower bound

$$N_4(2) \geq 432 \ .$$

The construction does not yield the equations of the quartic surfaces in our two-dimensional family. It would be interesting to find these equations.

Convention. Throughout this note the base field is $\mathbb{C}$.

1. Quartic surfaces with 16 skew conics and abelian surfaces of type (1, 9)

In this section we describe the relation between smooth quartic surfaces in $\mathbb{P}_3$ containing 16 mutually disjoint conics and polarized abelian surfaces of type $(1,9)$.

Let A be an abelian surface. We denote by $\sigma : \widetilde{A} \longrightarrow A$ the blow-up of A in the sixteen halfperiods $e_1, \dots, e_{16}$ and by $E_1, \dots, E_{16} \subset \widetilde{A}$ the exceptional curves. Let $\pi : \widetilde{A} \longrightarrow X$ be the projection onto the smooth Kummer surface X of A, i.e. onto the quotient of $\widetilde{A}$ by the involution induced by $a \longrightarrow -a$, $a \in A$. The exceptional curves $E_1, \dots, E_{16}$ map to (-2)-curves $D_1, \dots, D_{16}$ on X. We have a commutative diagram

$$\begin{array}{ccccccc} E_i & \subset & \widetilde{A} & \xrightarrow{\sigma} & A & \ni e_i \\ \downarrow & & \pi\downarrow & & \downarrow & \\ D_i & \subset & X & \longrightarrow & K & \end{array}$$

where K is the (singular) Kummer surface of A.

Now let $\mathcal{L}$ be an ample symmetric line bundle on A. We will be particularly interested in the linear system $|2\mathcal{L} \otimes \prod_{i=1}^{16} \mathcal{I}_{e_i}^2|^+$ of even divisors of the totally symmetric line bundle $2\mathcal{L}$ having multiplicity at least 2 in the sixteen halfperiods. This noncomplete linear system on A corresponds to a complete linear system on X. More precisely, there is a line bundle $\mathcal{M}_{\mathcal{L}}$ on X such that

$$\pi^*\mathcal{M}_{\mathcal{L}} = 2\sigma^*\mathcal{L} - 2\sum_{i=1}^{16} E_i H^0(\mathcal{M}_{\mathcal{L}}) \cong H^0(2\mathcal{L} \otimes \prod_{i=1}^{16} \mathcal{I}_{e_i}^2)^+ \ .$$

The basic observation underlying the approach of [1] is the following

Proposition 1.1 *For a smooth surface X the following conditions are equivalent:*

i) *X can be embedded into $\mathbb{P}_3$ as a quartic surface with 16 skew conics.*

ii) *X is the smooth Kummer surface of a polarized abelian surface $(A, \mathcal{L})$ of type $(1,9)$ and the line bundle $\mathcal{M}_{\mathcal{L}}$ on X is very ample.*

Proof. As for ii)$\Longrightarrow$i): By Kodaira vanishing $\mathcal{M}_\mathcal{L}$ has no higher cohomology, so we get $h^0(\mathcal{M}_\mathcal{L}) = \chi(\mathcal{M}_\mathcal{L}) = 4$ by Riemann-Roch. Thus $\mathcal{M}_\mathcal{L}$ defines an embedding $X \hookrightarrow \mathbb{P}_3$. Because of $\mathcal{M}_\mathcal{L} D_j = (2\sigma^*\mathcal{L} - 2\sum_{i=1}^{16} E_i)E_j = 2$ for $1 \le j \le 16$ the (-2)-curves $D_1, \dots, D_{16}$ map to skew conics.

As for i)$\Longrightarrow$ii): According to Nikulin's theorem [7] the quartic X is the smooth Kummer surface of an abelian surface A, the 16 skew conics being the (-2)-curves $D_1, \dots, D_{16}$ corresponding to the halfperiods on A. We have $(\pi^*\mathcal{O}_X(1) + 2\sum_{i=1}^{16} E_i)E_j = 0$ for $j = 1, \dots, 16$, so the line bundle $\pi^*\mathcal{O}_X(1) + 2\sum E_i$ descends to a line bundle on A, which is totally symmetric, hence of the form $2\mathcal{L}$ for some symmetric line bundle $\mathcal{L}$. Since $\mathcal{L}$ is effective and

$$\mathcal{L}^2 = (\tfrac{1}{2}\pi^*\mathcal{O}_X(1) + \sum E_i)^2 = 18$$

the bundle $\mathcal{L}$ is ample of type $(1,9)$ or $(3,3)$. Assume the latter case. Then $\mathcal{L} = \mathcal{O}_A(3\Theta)$ where Θ is an effective symmetric divisor of type $(1,1)$. Let $\widehat{\Theta}$ be its proper transform on $\widetilde{A}$. Since Θ contains at least six halfperiods, we find

$$\mathcal{O}_X(1) \cdot \pi(\widehat{\Theta}) = \mathcal{L}\Theta - \sum_{i=1}^{16} \operatorname{mult}_{e_i}(\Theta) \le 0 \ ,$$

a contradiction. So $\mathcal{L}$ is in fact of type $(1,9)$.

Finally, we have $\pi^*\mathcal{O}_X(1) = \pi^*\mathcal{M}_\mathcal{L}$, so $\mathcal{M}_\mathcal{L}$ is numerically equivalent – hence isomorphic – to $\mathcal{O}_X(1)$. □

2. The classification

It was shown in [1] that for a *generic* polarized abelian surface $(A, \mathcal{L})$ of type $(1,9)$ the line bundle $\mathcal{M}_\mathcal{L}$ is in fact very ample. In this section we will determine the precise conditions on $\mathcal{L}$ under which this happens. We prove:

Theorem 2.1 *Let A be an abelian surface and let $\mathcal{L}$ be an ample symmetric line bundle of type $(1,9)$ on A. Then we have:*

a) *$\mathcal{M}_\mathcal{L}$ fails to be ample if and only if the polarization $\mathcal{L}$ splits in one of the following four ways:*

I) *$\mathcal{L} = \mathcal{O}_A(E_1 + 9E_2)$, where E_1 and E_2 are elliptic curves with $E_1E_2 = 1$.*

II) *$\mathcal{L} = \mathcal{O}_A(P + 4E)$, where P is a divisor defining a principal polarization and E is an elliptic curve with $PE = 2$.*

III) *$\mathcal{L} = \mathcal{O}_A(E_1 + 3E_2)$, where E_1 and E_2 are elliptic curves with $E_1E_2 = 3$.*

IV) *$\mathcal{L} = \mathcal{O}_A(P + 2E)$, where P is a divisor defining a principal polarization and E is an elliptic curve with $PE = 4$.*

b) *Assume that $\mathcal{M}_\mathcal{L}$ is ample. Then it fails to be very ample if and only if $\mathcal{L}$ is of the form $\mathcal{L} = \mathcal{O}_A(P_1 + P_2)$, where P_1 and P_2 are ample divisors of type $(1,2)$ with $P_1P_2 = 5$.*

First we recall from [1, Sect. 1] how the intersection numbers of curves on the smooth Kummer surface X relate to those of their symmetric preimages on the abelian surface A. Here and in the sequel we abbreviate $\mathcal{M} = \mathcal{M}_{\mathcal{L}}$.

Lemma 2.2 *Let $C \subset X$ be an irreducible curve, different from $D_1, \dots, D_{16}$, and let $F = \sigma\pi^*C$ be the corresponding symmetric curve on A. Further, let $m_i = \mathrm{mult}_{e_i}(F)$ for $1 \le i \le 16$. Then we have:*
a) $F^2 = 2C^2 + \sum_{i=1}^{16} m_i^2$
b) $\mathcal{L}F = \mathcal{M}C + \sum_{i=1}^{16} m_i$

Now we show:

Proposition 2.3 *$\mathcal{M}$ is ample except in the cases I)-IV) of the theorem.*

Proof. Assume that $\mathcal{M}$ is not ample. Then there is an irreducible curve $C \subset X$ with $\mathcal{M}C \le 0$. If C is such a curve, then $C^2 = -2$ and C is different from $D_1, \dots, D_{16}$. We consider its symmetric preimage $F := \sigma\pi^*C$ on A. It satisfies

$$\mathcal{L}F \le \sum_{i=1}^{16} m_i F^2 = -4 + \sum_{i=1}^{16} m_i^2$$

where the m_i are the multiplicities of F in the halfperiods e_i of A.

We claim that $m_i \le 2$ for $1 \le i \le 16$. In order to prove this, first suppose $\mathcal{M}C = 0$. Applying the Hodge inequality to $\mathcal{M}$ and $C + D_i$ we find

$$4(-4 + 2CD_i) = \mathcal{M}^2(C + D_i)^2 \le (\mathcal{M}(C + D_i))^2 = 4 \ ,$$

hence $m_i = CD_i \le 2$. Now suppose $\mathcal{M}C < 0$. So C is a fixed component of $|\mathcal{M}|$. If D_i is fixed in $|\mathcal{M}|$ as well, then we must have $h^0(C + D_i) = 1$. We may assume $CD_i > 0$, i.e. that $C + D_i$ is connected. Then $h^1(C + D_i) = h^0(\mathcal{O}_{C+D_i}) - 1 = 0$ and Riemann-Roch gives

$$h^0(C + D_i) = 2 + \frac{1}{2}(C + D_i)^2 \ ,$$

implying $m_i = CD_i = 1$. On the other hand, if D_i is not fixed in $|\mathcal{M}|$, then it is not fixed in $|\mathcal{M} - C|$ either, hence $(\mathcal{M} - C)D_i \ge 0$, which implies $m_i = CD_i \le 2$. So the claim is proved.

For $k = 1, 2$ we denote by n_k the number of subscripts i, $1 \le i \le 16$, such that $m_i = k$. Then we have

$$F\mathcal{L} \le n_1 + 2n_2 F^2 = -4 + n_1 + 4n_2 \ .$$

Next we apply the Hodge inequality to $\mathcal{L}$ and F to obtain

$$18(-4 + n_1 + 4n_2) = \mathcal{L}^2F^2 \le (\mathcal{L}F)^2 \le (n_1 + 2n_2)^2 \tag{1}$$

Since n_1 is the number of odd halfperiods of the symmetric line bundle $\mathcal{O}_A(F)$, by [6, Proposition 4.7.5] it can only take the values 0,4,6,8,10,12,16. Using $n_1+n_2 \le 16$ and (1) we find the following list of possible values for n_1, n_2:

n_1	n_2	F^2	$\mathcal{L}F$
0	0	-4	0
0	1	0	≤ 2
4	0	0	≤ 4
6	0	2	6
12	0	8	12
16	0	12	≤ 16

Obviously $(n_1, n_2) = (0,0)$ is impossible. If $(n_1, n_2) = (0,1)$, then $F^2 = 0$ implies that F is a sum of two algebraically equivalent elliptic curves. But a symmetric elliptic curve contains either four halfperiods or none at all. In the cases $(n_1, n_2) = (6,0)$ and $(n_1, n_2) = (12,0)$ the Hodge index theorem implies that $\mathcal{L}$ and F are proportional: $\mathcal{L} \equiv_{\text{alg}} 3F$ resp. $2\mathcal{L} \equiv_{\text{alg}} 3F$. But this is impossible, because the line bundle $\mathcal{L}$ is primitive. In case $(n_1, n_2) = (16,0)$ the bundle $\mathcal{O}_A(F)$ is totally symmetric. But this is impossible, because then F^2 would have to be divisible by 8.

After all we see that we must have $(n_1, n_2) = (4,0)$, F being an elliptic curve and $\mathcal{L}F \le 4$.

If $\mathcal{L}F = 1$, then $\mathcal{L}$ is a product polarization, so we are in Case I).

In case $\mathcal{L}F = 2$ we consider the line bundle $\mathcal{L} - 4F$. Because of

$$(\mathcal{L} - 4F)^2 = 2, \quad (\mathcal{L} - 4F)\mathcal{L} = 10$$

it is ample of type $(1,1)$. Denoting by P the divisor in $|\mathcal{L} - 4F|$ we obtain

$$\mathcal{L} = \mathcal{O}_A(P + 4F) PF = 2$$

so that we are in Case II).

If $\mathcal{L}F = 3$, then we have

$$(\mathcal{L} - 3F)^2 = 0, \quad (\mathcal{L} - 3F)\mathcal{L} = 9 \ .$$

According to Lemma 2.4 below the bundle $\mathcal{L} - 3F$ is effective. Let E be a divisor in $|\mathcal{L} - 3F|$. Then we have $\mathcal{L} = \mathcal{O}_A(E + 3F)$ and $EF = 3$. The divisor E is either an elliptic curve or a sum of three algebraically equivalent elliptic curves. The latter however cannot happen, since then $\mathcal{L}$ would be of type $(3,3)$. We conclude that we are in Case III).

Finally, if $\mathcal{L}F = 4$, we find

$$(\mathcal{L} - 2F)^2 = 2, \quad (\mathcal{L} - 2F)\mathcal{L} = 10$$

implying that we are in Case IV).

Conversely, if the polarization $\mathcal{L}$ is of one of the types I)-IV), then there is an elliptic curve on A whose image C in X satisfies $\mathcal{M}C \le 0$, hence $\mathcal{M}$ is not ample.

$\square$

Lemma 2.4 *Let L be a line bundle on an abelian surface A such that $L^2 = 0$ and $LL_0 > 0$ for some ample line bundle L_0. Then L is effective.*

Proof. Let $K(L)$ be the kernel of the homomorphism $A \longrightarrow \widehat{A}$, $a \longmapsto t_a^*L - L$ and let E be the connected component of $K(L)$ containing the origin of A. The assumptions on L imply that the hermitian form of L is of rank 1, hence E is an elliptic curve. We have $K(L|E) = E$, thus $L|E \in \mathrm{Pic}^0(E)$ by [6, Lemma 2.4.7]. Replacing L by a suitable translate we may therefore assume that $L|E$ is trivial. According to [6, Lemma 3.3.2] then L descends to a line bundle $\overline{L}$ on the elliptic curve X/E and $H^0(L) \cong H^0(\overline{L})$. Because of $LL_0 > 0$ the line bundle $\overline{L}$ must be of positive degree, hence $h^0(L) = h^0(\overline{L}) > 0$. □

Next we show:

Proposition 2.5 *If $\mathcal{M}$ is ample, then it is globally generated.*

Proof. Assume that the linear system $|\mathcal{M}|$ has base points. Since $\mathcal{M}$ is ample, it follows from [8, Proposition 8.1] that $\mathcal{M}$ is of the form

$$\mathcal{M} = \mathcal{O}_X(kE + \Gamma)$$

where $E, \Gamma \subset X$ are irreducible curves such that $E^2 = 0$, $\Gamma^2 = -2$, $E\Gamma = 1$ and

$$|\mathcal{M}| = |kE| + \Gamma = |E| + \ldots + |E| + \Gamma \ .$$

We have $4 = \mathcal{M}^2 = 2k - 2$, hence $k = 3$. Since the linear system $|2\mathcal{L} - 2\sum e_i|^+$ on A is invariant under translation by halfperiods, the base part Γ of $\mathcal{M}$ cannot be one of the curves $D_1, \ldots, D_{16}$. Therefore we have $\Gamma D_i \geq 0$, thus

$$2 = \mathcal{M}D_i = 3ED_i + \Gamma D_i$$

implies $ED_i = 0$. We consider the symmetric preimage $G \subset A$ of the elliptic curve $E \subset X$. Because of $ED_i = 0$ it must be a sum of two algebraically equivalent elliptic curves. But this is a contradiction with

$$\mathcal{L}G = \mathcal{M}E + \sum m_i = \Gamma E + \sum ED_i = 1 \ .$$

□

Now we complete the proof of Theorem 2.1 by

Proposition 2.6 *Suppose that $\mathcal{M}$ is ample. Then $\mathcal{M}$ is very ample if and only if $\mathcal{L}$ is not of the form*

$$\mathcal{L} = \mathcal{O}_A(P_1 + P_2)$$

where P_1, P_2 are ample divisors of type $(1, 2)$ with $P_1P_2 = 5$.

Proof. According to [8, Theorem 5.2 and Theorem 6.1.iii] $\mathcal{M}$ is very ample if and only if

i) there is no elliptic curve $C \subset X$ with $\mathcal{M}C = 2$, and

ii) there is no irreducible curve $H \subset X$ with $H^2 = 2$ and $\mathcal{M} = \mathcal{O}_X(2H)$.

Because of $\mathcal{M}^2 = 4$ condition ii) is certainly fulfilled. Now assume that there is a curve $C \subset X$ as in i). Then we have

$$\mathcal{L}F = 2 + \sum m_i F^2 = \sum m_i^2$$

for its symmetric preimage $F \subset A$. The Hodge inequality for $\mathcal{M}$ and $2C + D_i$ gives

$$4(2C + D_i)^2 = \mathcal{M}^2(2C + D_i)^2 \leq (\mathcal{M}(2C + D_i))^2 = 36 \ ,$$

thus $m_i = CD_i \leq 2$. So we can write

$$\mathcal{L}F = 2 + n_1 + 2n_2 F^2 = n_1 + 4n_2$$

with n_1, n_2 defined as before. Using $n_1 + n_2 \leq 16$ and

$$18(n_1 + 4n_2) = \mathcal{L}^2 F^2 \leq (\mathcal{L}F)^2 = (2 + n_1 + 2n_2)^2$$

we find that there are only the following 3 possibilities:

1) $n_1 = n_2 = 0$, $F^2 = 0$, $\mathcal{L}F = 2$

2) $n_1 = 0$, $n_2 = 16$, $F^2 = 64$, $\mathcal{L}F = 34$

3) $n_1 = 16$, $n_2 = 0$, $F^2 = 16$, $\mathcal{L}F = 18$

In Case 1) the curve F consists of two elliptic curves F_1, F_2 with $\mathcal{L}F_1 = \mathcal{L}F_2 = 1$. But then $\mathcal{L}$ would be a product polarization and $\mathcal{M}$ would not be ample by Proposition 2.3.

In Case 2) the line bundle $\mathcal{O}_A(F)$ is totally symmetric, hence it is the square of an ample line bundle $\mathcal{O}_A(F_0)$ of selfintersection 16. For the bundle $\mathcal{L} - F_0$ we have

$$(\mathcal{L} - F_0)^2 = 0, \quad (\mathcal{L} - F_0)\mathcal{L} = 1$$

so by Lemma 2.4 the system $|\mathcal{L} - F_0|$ contains an elliptic curve. But then $\mathcal{L}$ would again be a product polarization.

Finally we consider Case 3). Here $\mathcal{O}_A(F)$ is the square of an ample line bundle $\mathcal{O}_A(P_1)$ of type $(1, 2)$. Because of

$$(\mathcal{L} - P_1)^2 = 4, \quad (\mathcal{L} - P_1)\mathcal{L} = 9$$

there is an ample divisor $P_2 \in |\mathcal{L} - P_1|$ of type $(1, 2)$. So we arrive at $\mathcal{L} = \mathcal{O}_A(P_1 + P_2)$ with $P_1 P_2 = 5$.

Conversely, if $\mathcal{L} = \mathcal{O}_A(P_1 + P_2)$ with P_1, P_2 of type $(1, 2)$ and $P_1 P_2 = 5$, then by [2, Lemma 3.6.b] the linear system $|2P_1 - \sum e_i|^-$ is free, so by Bertini's theorem there is a smooth curve in this system. Its image C in X is an elliptic curve with $\mathcal{M}C = 2$, hence $\mathcal{M}$ is not very ample. □

3. Quartic surfaces with 432 conics

According to [1] the smooth Kummer surface of a *generic* abelian surface of type $(1,9)$ embeds into $\mathbb{P}_3$ as a quartic surface with 352 conics. In this section we will show that by specializing to abelian surfaces with real multiplication one can obtain quartic surfaces with still more conics on them.

Let $(A, \mathcal{L}_0)$ be a principally polarized abelian surface such that the endomorphism ring of A is isomorphic to $\mathbb{Z}[\sqrt{7}]$. According to [3, Proposition 2.1] there is a 2-dimensional family of such surfaces. The endomorphism associated to $4+\sqrt{7}$ has minimal polynomial $t^2 - 8t + 9$, hence by [6, Proposition 2.3] it determines a polarization $\mathcal{L}$ with $\mathcal{L}\mathcal{L}_0 = 8$ and $\mathcal{L}^2 = 18$. This implies that $\mathcal{L}$ is of type $(1,9)$. The Néron-Severi group $\mathrm{NS}(A)$ is generated by $\mathcal{L}_0$ and $\mathcal{L}$. We may assume that the line bundles $\mathcal{L}_0$ and $\mathcal{L}$ are symmetric.

As before we denote by X the smooth Kummer surface of A and by $\mathcal{M} = \mathcal{M}_{\mathcal{L}}$ the line bundle on X associated to the linear system $|2\mathcal{L} \otimes \prod_{i=1}^{16} \mathcal{I}_{e_i}^2|^+$ on A.

We begin by showing:

Proposition 3.1 *$\mathcal{M}$ is very ample.*

Proof. Assume the contrary. According to Theorem 2.1 then A is not simple or A admits a polarization P of type $(1,2)$ such that $\mathcal{L}P = 9$.

First suppose that A contains an elliptic curve E. We have $E \equiv_{\mathrm{alg}} n\mathcal{L} + m\mathcal{L}_0$ for some $m, n \in \mathbb{Z}$, hence

$$0 = E^2/2 = 9n^2 + 8mn + m^2 \ . \tag{2}$$

We may assume that not both m and n are even. Then (2) shows that they both must be odd, which implies $m^2, n^2 \equiv 1 \bmod 8$. But then $m^2 + n^2 \equiv 2 \neq 0 \bmod 8$, a contradiction with (2).

Now assume that there is a polarization P of type $(1,2)$ on A with $\mathcal{L}P = 9$. Writing $P \equiv_{\mathrm{alg}} n\mathcal{L} + m\mathcal{L}_0$ with $m, n \in \mathbb{Z}$ we have

$$9 = \mathcal{L}P = 18n + 8m \equiv 0 \bmod 2 \ ,$$

a contradiction. $\square$

According to the previous proposition we may consider X as a smooth quartic surface in $\mathbb{P}_3$, the bundle $\mathcal{M}$ being the hyperplane bundle $\mathcal{O}_X(1)$. We show:

Proposition 3.2 *The quartic surface X contains exactly 432 smooth conics.*

Proof. 1) As in the generic case (see [1]) we have 352 conics on X arising as

- the 16 images $D_1, \dots, D_{16}$ of the exceptional curves $E_1, \dots, E_{16}$ of $\widetilde{A}$,
- the 16 conics $C_1, \dots, C_{16}$, which are complementary to $D_1, \dots, D_{16}$, i.e. $D_i + C_i \in |\mathcal{M}|$. These are the images of symmetric divisors in translates of $|2\mathcal{L}|$, vanishing in e_i to the fourth order and in the remaining 15 halfperiods to the second order.

• $20 \cdot 16$ conics C^i_{klm}, which are the images of the divisors in $|t^*_{e_i}\mathcal{L}|^-$ passing through three odd halfperiods e_k, e_l, e_m of $t^*_{e_i}\mathcal{L}$, $1 \le i \le 16$, $1 \le k < l < m \le 6$.

2) The principal polarization $\mathcal{L}_0$ provides us with 32 conics, namely
• the images of the 16 symmetric translates of the unique divisor in $|\mathcal{L}_0|$, and
• the 16 conics, which are complementary to them. They are the images of symmetric divisors in translates of $|2\mathcal{L} - \mathcal{L}_0|$ vanishing in 6 halfperiods to the first order and in 10 halfperiods to the second order.

3) Next we show that there are 48 conics on X coming from certain translates of the line bundle $P := \mathcal{L} - \mathcal{L}_0$. First note that because of

$$\begin{aligned} P^2 &= (\mathcal{L} - \mathcal{L}_0)^2 = 4 \\ P\mathcal{L}_0 &= (\mathcal{L} - \mathcal{L}_0)\mathcal{L}_0 > 0 \end{aligned}$$

the bundle P is ample of type $(1,2)$. Let, as usual, $K(P)$ denote the kernel of the isogeny

$$\begin{aligned} A &\longrightarrow \widehat{A} \\ a &\longmapsto t^*_a P - P \ , \end{aligned}$$

where $\widehat{A} = \mathrm{Pic}^0(A)$ is the dual abelian surface. Then the 16 symmetric translates of P are the line bundles $t^*_a P$, where a runs through a system of representatives for the factor group

$$\frac{1}{2}K(P)/K(P)$$

The number of odd/even halfperiods of a translate $t^*_a P$ is either 8/8 or 4/12. Using [4, Theorem 5.4] one finds that the translates of the 4/12-type are in bijective correspondence with the representatives a of $\frac{1}{2}K(P)/K(P)$ satisfying

$$2a \in 2K(P) \ . \tag{3}$$

Since P is of type $(1,2)$, we have $K(P) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, hence $2K(P) = 0$. So condition (3) simply means that a is a halfperiod. Modulo $K(P)$ there are exactly four halfperiods, so four of the sixteen translates of P are of type 4/12, whereas twelve are of type 8/8. If $t^*_a P$ is of the latter type, then $h^0(t^*_a P)^\pm = 1$. The divisors $F^\pm \in |t^*_a P|^\pm$ contain 8 halfperiods to an odd order. They map to smooth conics $C^\pm$ in X, because

$$10 = \mathcal{L}F^\pm = \mathcal{M}C^\pm + \sum m_i$$

and the ampleness of $\mathcal{M}$ imply $\sum m_i = 8$ and $\mathcal{M}C^\pm = 2$. So we obtain
• $2 \cdot 12$ conics from the 12 translates in question, and
• $2 \cdot 12$ conics complementary to these, which correspond to symmetric divisors in 12 translates of $|\mathcal{L} + \mathcal{L}_0|$ vanishing in 8 halfperiods to the first order and in the other 8 halfperiods to the second order.

4) It remains to show that the 352+32+48=432 conics considered so far are the only ones on X. So let $C \subset X$ be any smooth conic. If C is different from $D_1, \ldots, D_{16}$ and from $C_1, \ldots, C_{16}$, then $0 \leq CD_i \leq 2$ for $i = 1, \ldots, 16$. So its symmetric preimage F on A satisfies

$$\mathcal{L}F = 2 + n_1 + 2n_2 F^2 = -4 + n_1 + 4n_2 \ ,$$

where n_1 and n_2 are the number of halfperiods, which F contains to the first resp. second order. On the other hand, writing $F \equiv_{\mathrm{alg}} n\mathcal{L} + m\mathcal{L}_0$ with $m, n \in \mathbb{Z}$ we find

$$\mathcal{L}F = 18n + 8mF^2 = 18n^2 + 16mn + 2m^2 \tag{4}$$

The Hodge inequality for $\mathcal{L}$ and F and the inequality $n_1 + n_2 \leq 16$ allow only finitely many values for $\mathcal{L}F$ and F^2. One finds that the integral solutions of (4) are given by

n_1	n_2	n	m
10	3	1	0
8	8	1	1
8	0	1	-1
6	10	2	-1
6	0	0	1

But then it is clear that C is one of the conics coming from $\mathcal{L}$, $\mathcal{L} + \mathcal{L}_0$, $\mathcal{L} - \mathcal{L}_0$, $2\mathcal{L} - \mathcal{L}_0$ and $\mathcal{L}_0$ respectively. $\square$

Note that there are no lines on the surface X. In fact, if we assume that X contains a line C, then $F := \sigma\pi^*C$ satisfies $\mathcal{L}F = 1 + n_1 + 2n_2$ with n_1, n_2 defined as before, thus $\mathcal{L}F$ is odd. On the other hand, writing $F \equiv_{\mathrm{alg}} n\mathcal{L} + m\mathcal{L}_0$ with $m, n \in \mathbb{Z}$ we find that $\mathcal{L}F = 18n + 8m$ is even, a contradiction.

Altogether we have shown in this section:

Theorem 3.3 *Let A be a principally polarized abelian surface with endomorphism ring isomorphic to $\mathbb{Z}[\sqrt{7}]$. Then the smooth Kummer surface of A embeds into $\mathbb{P}_3$ as a smooth quartic. There are no lines and exactly 432 smooth conics on this quartic surface.*

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