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On tensor products of ample line bundles on abelian varieties

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0. Introduction

In the present note we consider the question of global generation and very ampleness of tensor products of ample line bundles on abelian varieties.

It is a classical result of Lefschetz ([3], Theorem 4.5.1) that the third tensor power of an ample line bundle on an abelian variety is very ample. It is also well known that the second power is globally generated. We generalize these results to the product of three resp. two arbitrary ample line bundles. Further we give criteria for a tensor product of two ample line bundles to be very ample.

The essential tool in the proof of the results for the powers of an ample line bundle is the theorem of the square ([3], Theorem 2.3.3). Our method is to replace this theorem by arguments involving the isogeny associated to an ample line bundle.

Throughout this note the base field is $\mathbb{C}$.

1. Very ample tensor products of three factors

Theorem 1.1 *Let X be an abelian variety and let L_1, L_2, L_3 be ample line bundles on X. Then we have:*
a) $L_1 + L_2$ is globally generated.
b) $L_1 + L_2 + L_3$ is very ample.

Proof. a) Let $a \in X$ and let $\Theta_1 \in |L_1|$, $\Theta_2 \in |L_2|$. Consider the homomorphism

$$\begin{aligned} \Phi : X \times X &\longrightarrow \operatorname{Pic}^0(X) \\ (x_1, x_2) &\longmapsto t_{x_1}^* L_1 - L_1 + t_{x_2}^* L_2 - L_2 \ . \end{aligned}$$

Here t_{x_i} denotes the translation map $X \longrightarrow X$, $x \longmapsto x + x_i$. Since L_1 and L_2 are ample, Φ is surjective, hence its kernel is of dimension $n := \dim X$. For any

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$x_1, x_2 \in X$ the intersections

$$\begin{aligned} \ker\Phi &\cap (\{x_1\}\times X) \\ \ker\Phi &\cap (X\times\{x_2\}) \end{aligned}$$

are finite, because the maps $\Phi(x_1,\cdot)$ and $\Phi(\cdot,x_2)$ are translates of the isogenies associated to the ample line bundles L_1, L_2. Therefore the intersections

$$\begin{aligned} \ker\Phi &\cap (t_a^*\Theta_1\times X) \\ \ker\Phi &\cap (X\times t_a^*\Theta_2) \end{aligned}$$

are of dimension $n-1$. Thus we see that there is a pair $(x_1,x_2)\in\ker\Phi$ such that

$$(x_1,x_2)\notin(t_a^*\Theta_1\times X)+(X\times t_a^*\Theta_2)\ .$$

Then we have

$$\begin{aligned} x_1 &\notin t_a^*\Theta_1 \quad \text{i.e.} \quad a\notin t_{x_1}^*\Theta_1 \\ x_2 &\notin t_a^*\Theta_2 \quad \text{i.e.} \quad a\notin t_{x_2}^*\Theta_2 \end{aligned}$$

and

$$t_{x_1}^*L_1+t_{x_2}^*L_2=L_1+L_2\ .$$

Consequently $t_{x_1}^*\Theta_1+t_{x_2}^*\Theta_2$ is a divisor in $|L_1+L_2|$, which does not contain the point a.

b) Let $L=L_1+L_2+L_3$ and let φ_L be the map defined by L. By a) φ_L is a morphism. We have to show that φ_L and the differential $d\varphi_L$ are both injective.

Step I. φ_L is injective.

Let $y_1,y_2\in X$ such that $\varphi_L(y_1)=\varphi_L(y_2)$. There is a reduced divisor $\Theta_1\in|L_1|$ such that Θ_1 coincides with none of its translates ([3], Proposition 4.1.7 and Lemma 4.1.8). Let x_1 be an arbitrary point of $t_{y_1}^*\Theta_1$. The line bundle $L-t_{x_1}^*\Theta_1$ is algebraically equivalent to L_2+L_3, hence globally generated by a). Therefore we find a divisor

$$\Theta'\in|L-t_{x_1}^*\Theta_1|$$

such that $y_2\notin\Theta'$. The divisor

$$t_{x_1}^*\Theta_1+\Theta'\in|L|$$

contains y_1, hence by assumption it also contains y_2. By choice of Θ' we must have $y_2\in t_{x_1}^*\Theta_1$, or equivalently $x_1\in t_{y_2}^*\Theta_1$. Since Θ_1 is reduced, we conclude

$$t_{y_1}^*\Theta_1\subset t_{y_2}^*\Theta_1\ .$$

Then we necessarily have

$$t_{y_1-y_2}^*\Theta_1=\Theta_1\ ,$$

hence by choice of Θ_1 it follows that $y_1 = y_2$.

Step II. $d\varphi_L$ is injective.

Suppose to the contrary that there exists a point $a \in X$ and a tangent vector T such that T is tangent to all the divisors $D \in |L|$ containing the point a. Let $\Theta_1 \in |L_1|$ be reduced and let $x_1 \in t_a^*\Theta_1$. As before there is a divisor $\Theta' \in |L - t_{x_1}^*\Theta_1|$ such that $a \notin \Theta'$. By assumption T is tangent to $\Theta' + t_{x_1}^*\Theta_1$, hence T must be tangent to $t_{x_1}^*\Theta_1$ in a, or equivalently T is tangent to Θ_1 in $a + x_1$. Since this holds for any $x_1 \in t_a^*\Theta_1$, we found that T is tangent to Θ_1 in all points of Θ_1. But then the image of the Gauß-map associated to Θ_1 is contained in a hyperplane, contradicting [3], Proposition 4.4.1. □

2. Tensor products of two ample line bundles

Now we want to give criteria for a tensor product of two ample line bundles to be very ample. This is easy in the surface case:

Theorem 2.1 *Let X be an abelian surface and let L_1, L_2 be ample line bundles, which are not algebraically equivalent. Then $L_1 + L_2$ is very ample, unless*

$$\begin{aligned} L_1 &\equiv_{\text{alg}} \mathcal{O}_X(E_1 + k_1 F) \\ L_2 &\equiv_{\text{alg}} \mathcal{O}_X(E_2 + k_2 F) \ , \end{aligned}$$

where k_1, k_2 are positive integers and $E_1, E_2, F \subset X$ are elliptic curves with $E_1 F = E_2 F = 1$.

Proof. Let $L = L_1 + L_2$. We have $L_1 L_2 \geq 3$, because otherwise the Hodge inequality would imply that L_1 and L_2 are algebraically equivalent. Therefore

$$L^2 = L_1^2 + L_2^2 + 6 \geq 10$$

and we may apply Reider's theorem [5]. So if L is not very ample, then there is an effective divisor E on X such that

$$LE \leq 2 \quad \text{and} \quad E^2 = 0 \ .$$

Then we must have $L_1 E = L_2 E = 1$ and E must be an elliptic curve. Considering the decomposition of divisors in $|L_i|$ into irreducible components we see that L_1 and L_2 are of the asserted forms. □

Next we consider the problem in arbitrary dimension. Let L_1, L_2 be ample line bundles on an abelian variety and let

$$\Phi : X \times X \longrightarrow \mathrm{Pic}^0(X)$$

be the homomorphism defined in the proof of Theorem 1.1. Let $p, q : \ker\Phi \longrightarrow X$ be the projections onto the first resp. second factor.

$$\begin{array}{ccc} & X\times X & \\ & \cup & \\ & \ker\Phi & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & & X \end{array}$$

The following theorem was proved in the case $L_1 = L_2$ by Ohbuchi [4].

Theorem 2.2 *Let X be an abelian variety and let L_1, L_2 be ample line bundles on X such that $|L_1|$ and $|L_2|$ have no base components. Then $L_1 + L_2$ is very ample.*

Proof. Let $L = L_1 + L_2$ and let φ_L be the rational map defined by L. According to Theorem 1.1 φ_L is a morphism.

Step I. φ_L is injective.

Let $y_1, y_2 \in X$ with $\varphi_L(y_1) = \varphi_L(y_2)$. There is an irreducible divisor $\Theta_1 \in |L_1|$ such that Θ_1 coincides with none of its proper translates ([3], Lemma 4.1.8 and Theorem 4.3.5). We may assume that $t^*_{y_1}\Theta_1$ and $t^*_{y_2}\Theta_1$ intersect properly, because otherwise we would have $t^*_{y_1}\Theta_1 = t^*_{y_2}\Theta_1$, hence $y_1 = y_2$ and we were done.

Let $x_1 \in t^*_{y_1}\Theta_1, x_1 \notin t^*_{y_2}\Theta_1$. Then

$$y_1 \in t^*_{x_1}\Theta_1 + t^*_{x_2}\Theta_2 \in |L|$$

for all $\Theta_2 \in |L_2|$ and all $x_2 \in qp^{-1}(x_1)$. By assumption on y_1 and y_2 we then have

$$y_2 \in t^*_{x_1}\Theta_1 + t^*_{x_2}\Theta_2$$

for all these Θ_2 and x_2. By choice of x_1 then

$$y_2 \in t^*_{x_2}\Theta_2 \ .$$

Now choose a divisor $\Theta_1' \neq \Theta_1 \in |L_1|$. Then we have

$$y_2 \in t^*_{x_2}\Theta_2 + t^*_{x_1}\Theta_1' \in |L| \ ,$$

hence

$$y_1 \in t^*_{x_2}\Theta_2 + t^*_{x_1}\Theta_1'$$

for all $\Theta_2 \in |L_2|$ and all $x_2 \in qp^{-1}(x_1)$. Since $\Theta_1' \neq \Theta_1$, the divisors $t^*_{y_1}\Theta_1'$ and $t^*_{y_1}\Theta_1$ meet properly. So for $x_1 \in t^*_{y_1}\Theta_1, x_1 \notin t^*_{y_2}\Theta_1, x_1 \notin t^*_{y_1}\Theta_1'$ we obtain

$$y_1 \in t^*_{x_2}\Theta_2 \ , \text{ thus } x_2 \in t^*_{y_1}\Theta_2$$

for all $x_2 \in qp^{-1}(x_1)$ and all $\Theta_2 \in |L_2|$. We conclude that

$$qp^{-1}(t^*_{y_1}\Theta_1) \subset t^*_{y_1}\Theta_2$$

for all $\Theta_2 \in |L_2|$. But this means that $|t^*_{y_1}L_2|$ has a base component—a contradiction.

Step II. $d\varphi_L$ is injective.

Suppose to the contrary that there is a point $a \in X$ and a tangent vector T such that T is tangent in a to all divisors in $|L|$ containing a.

Choose $\Theta_1 \in |L_1|$ irreducible and let $x_1 \in t^*_a\Theta_1$ be a smooth point. Then

$$a \in t^*_{x_1}\Theta_1 + t^*_{x_2}\Theta_2 \in |L|$$

for all $\Theta_2 \in |L_2|$ and all $x_2 \in qp^{-1}(x_1)$.

Now let M be the subset of $t^*_a\Theta_1$ consisting of all points x_1 such that

$$x_2 \in t^*_a\Theta_2$$

holds for all $\Theta_2 \in |L_2|$ and all $x_2 \in qp^{-1}(x_1)$. Thus we have

$$qp^{-1}(M) \subset t^*_a\Theta_2 \quad \text{for all } \Theta_2 \in |L_2| \ .$$

Since $|t^*_aL_2|$ has no base components, $qp^{-1}(M)$ cannot contain an open set of dimension $\dim X - 1$. So $t^*_a\Theta_1 \setminus M$ is a dense subset of $t^*_a\Theta_1$, because p and q are finite maps. Now let $x_1 \in t^*_a\Theta_1 \setminus M$ be a smooth point. The condition $x_1 \notin M$ implies that T is tangent to $t^*_{x_1}\Theta_1$ in a. Equivalently, T is tangent to Θ_1 in $a + x_1$. Since this holds for a dense subset of points $x_1 \in t^*_a\Theta_1$, T is in fact tangent to Θ_1 in all of its points. But then the image of the Gauß-map of Θ_1 is contained in a hyperplane—contradicting the fact that L_1 is ample. □

Finally we consider tensor products of principal polarizations generalizing results for abelian surfaces and abelian threefolds obtained in [2] resp. [1] by the Comessatti-method.

Theorem 2.3 *Let L_1 and L_2 be principal polarizations on an abelian variety X such that L_1 is an irreducible polarization. Then $L_1 + L_2$ is very ample unless L_1 and L_2 are algebraically equivalent.*

Proof. If L_1 and L_2 are algebraically equivalent, then $L_1 + L_2$ is the Kummer polarization, hence certainly not very ample. So assume that L_1 and L_2 are not algebraically equivalent.

Step I. Let Θ_1, Θ_2 be the divisors in $|L_1|$ resp. $|L_2|$. First we show that

$$t^*_{y_1}\Theta_1 \not\subset pq^{-1}t^*_{y_2}\Theta_2 \tag{$*$}$$

for all $y_1, y_2 \in X$. Suppose the contrary. Since the homomorphism Φ only depends on the algebraic equivalence classes of L_1 and L_2, we may assume $y_1 = y_2 = 0$. L_1

and L_2 being principal polarizations the projections p and q are morphisms of degree 1. By Zariski's main theorem p and q are even isomorphisms. So the composed map $f := pq^{-1} : X \longrightarrow X$ is an isomorphism. Since Θ_1 is irreducible, we have the equality $pq^{-1}\Theta_2 = \Theta_1$, i.e. $\Theta_2 = f^*\Theta_1$. We have $f(0) = 0$, so f is a homomorphism of abelian varieties. Thus we have for any $x_1 \in X$

$$f^*(\Theta_1 - t_{x_1}^*\Theta_1) = f^*\Theta_1 - f^*t_{x_1}^*\Theta_1 = f^*\Theta_1 - t_{f^{-1}(x_1)}^*f^*\Theta_1 \ ,$$

which by assumption equals

$$\Theta_2 - t_{f^{-1}(x_1)}^*\Theta_2 \ .$$

By definition of f this divisor is linearly equivalent to

$$-(\Theta_1 - t_{x_1}^*\Theta_1) \ .$$

So we found that the dual morphism $\widehat{f} : \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(X)$ is just the map $M \longrightarrow -M$. Denoting the involution $X \longrightarrow X$, $x \longmapsto -x$ by ι this means

$$\widehat{f} = \widehat{\iota} \ ,$$

hence $f = \iota$ and we conclude $L_2 = \iota^* L_1$. But then L_1 and L_2 are algebraically equivalent—contradicting our assumption.

Now let $L = L_1 + L_2$ and let φ_L be the rational map defined by L. According to Theorem 1.1 φ_L is a morphism.

Step II. φ_L is injective.

Let $y_1, y_2 \in X$ with $\varphi_L(y_1) = \varphi_L(y_2)$. Let $x_1 \in t_{y_1}^*\Theta_1$ be an arbitrary point. Then

$$y_1 \in t_{x_1}^*\Theta_1 + t_{x_2}^*\Theta_2 \in |L|$$

for $x_2 = qp^{-1}(x_1)$. By assumption on y_1 and y_2 this implies

$$y_2 \in t_{x_1}^*\Theta_1 + t_{x_2}^*\Theta_2 \ .$$

If $y_2 \in t_{x_2}^*\Theta_2$, then $x_2 \in t_{y_2}^*\Theta_2$, hence $x_1 \in pq^{-1}t_{y_2}^*\Theta_2$. So we found

$$t_{y_1}^*\Theta_1 \subset t_{y_2}^*\Theta_1 + pq^{-1}t_{y_2}^*\Theta_2 \ ,$$

which implies $t_{y_1}^*\Theta_1 \subset t_{y_2}^*\Theta_1$ because of $(*)$. But then we have $t_{y_1-y_2}^*\Theta_1 = \Theta_1$, hence $y_1 = y_2$.

Step III. $d\varphi_L$ is injective.

Suppose that there is a point $a \in X$ and a tangent vector T which is tangent in a to all divisors in $|L|$ containing a. Let $x_1 \in t_a^*\Theta_1$ be a smooth point with $x_1 \notin pq^{-1}t_a^*\Theta_2$. Then

$$a \in t_{x_1}^*\Theta_1 + t_{x_2}^*\Theta_2 \in |L|$$

for $x_2 = qp^{-1}(x_1)$. The point a cannot be contained in $t^*_{x_2}\Theta_2$, because otherwise we would have $x_2 \in t^*_a\Theta_2$, hence $x_1 \in pq^{-1}t^*_a\Theta_2$. By assumption T is then tangent to $t^*_{x_1}\Theta_1$ in a. Now we can proceed as in the proof of Theorem 2.2 to obtain a contradiction. $\square$

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