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Higher order embeddings of abelian varieties

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0. Introduction

In recent years several concepts of higher order embeddings have been introduced and studied by Beltrametti, Francia, Sommese and others: k-spannedness, k-very ampleness and k-jet ampleness (see [BFS], [BeSo1], [BeSo2], [BeSo3]).

First recall the definitions:

Definition. Let X be a smooth projective variety and L a line bundle on X.

(a) L is called *k-very ample* (resp. *k-spanned*), if for any zero-dimensional subscheme $(Z, \mathcal{O}_Z)$ of X of length $k+1$ (resp. for any curvilinear zero-dimensional subscheme $(Z, \mathcal{O}_Z)$ of X of length $k+1$) the restriction map

$$H^0(L) \longrightarrow H^0\left(L \otimes \mathcal{O}_Z\right)$$

is surjective. Here a subscheme is called *curvilinear*, if it is locally contained in a smooth curve.

(b) L is called *k-jet ample*, if the restriction map

$$H^0(L) \longrightarrow H^0\left(L \otimes \mathcal{O}_X / \left(\mathfrak{m}_{y_1}^{k_1} \otimes \ldots \otimes \mathfrak{m}_{y_r}^{k_r}\right)\right)$$

is surjective for any choice of distinct points $y_1, \ldots, y_r$ in X and positive integers $k_1, \ldots, k_r$ with $\sum k_i = k+1$.

The strongest notion is k-jet ampleness; it implies k-very ampleness (cf. [BeSo2, Proposition 2.2]) which of course implies k-spannedness. For $k=0$ or $k=1$ all the three notions are equivalent and correspond to global generation resp. very ampleness.

In this note we give criteria for k-jet ampleness of line bundles on abelian varieties. A naive way to obtain such a criterion is as follows: According to [BeSo2, Corollary 2.1] a tensor product of k very ample line bundles is always k-jet ample. Now on an abelian variety, by the generalization of Lefschetz' classical theorem [LB,

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Theorem 4.5.1] given in [BaSz, Theorem 1.1], one knows that a tensor product of three ample line bundles is already very ample. So the conclusion is that *a tensor product of* $3k$ *ample line bundles on an abelian variety is* k*-jet ample.* In this note we show that actually the following considerably stronger statement holds:

Theorem 1. *Let* A *be an abelian variety and let* $L_1, \dots, L_{k+2}$ *be ample line bundles on* A*,* $k \geq 0$*. Then* $L_1 + \ldots + L_{k+2}$ *is* k*-jet ample.*

This result is sharp in the sense that in general a tensor product of only $k+1$ ample line bundles on an abelian variety is not k-spanned, thus not k-very ample or k-jet ample (see Proposition 2.4). However, it is an interesting problem to specify additional assumptions on $k+1$ ample line bundles, which ensure that their tensor product is k-jet ample.

Here we show:

Theorem 2. *Let* A *be an abelian variety and let* $L_1, \dots, L_{k+1}$ *be ample line bundles on* A*,* $k \geq 1$*. Assume that* L_{k+1} *has no fixed components. Then* $L_1 + \ldots + L_{k+1}$ *is* k*-jet ample.*

Actually Theorem 1 is a corollary of Theorem 2, due to the fact that a tensor product of two ample line bundles on an abelian varieties is always globally generated ([BaSz]).

Notation and Conventions. We work throughout over the field $\mathbb{C}$ of complex numbers.

For a point x on an abelian variety A we denote by $t_x : A \longrightarrow A$ the translation map $a \longmapsto a + x$. A divisor Θ on A is called *translation-free*, if $t_x^*\Theta = \Theta$ implies $x = 0$.

If L is a line bundle on A, $x \in A$ a point and $k \geq 0$ an integer, the map $H^0(L) \longrightarrow H^0\left(L \otimes \mathcal{O}_A/\mathfrak{m}_x^{k+1}\right)$ mapping a global section of L to its k-jet at x is denoted by $j^k_{L,x}$ or simply by j^k_x.

For a reduced divisor D we denote by $(D)_s$ its smooth part.

1. Higher order Gauß maps

Let A be an abelian variety and let D be a reduced divisor on A defined by a section $s \in H^0\left(\mathcal{O}_A(D)\right)$. The *Gauß map* of D is defined as

$$\begin{array}{rcl} \gamma_D : (D)_s & \longrightarrow & \mathbb{P}\left(\mathfrak{m}_0/\mathfrak{m}_0^2\right) \\ x & \longmapsto & \mathbb{C} \cdot \alpha_x\left(j^1_x(s)\right) \end{array}$$

where

$$\alpha_x : \mathbb{P}\left(\mathcal{O}_A(D) \otimes \mathfrak{m}_x/\mathfrak{m}_x^2\right) \longrightarrow \mathbb{P}\left(\mathfrak{m}_x/\mathfrak{m}_x^2\right) \longrightarrow \mathbb{P}\left(\mathfrak{m}_0/\mathfrak{m}_0^2\right)$$

is the canonical isomorphism (0 being the zero point on A). Identifying $\mathfrak{m}_0/\mathfrak{m}_0^2$ with the dual of the universal covering space of A, the map γ_D coincides with the Gauß map of D defined in [LB, Section 4.4].

Next, let $D_1, \ldots, D_n$ be reduced divisors on A, defined by sections $s_1, \ldots, s_n$ respectively. We define the *n-th order Gauß map of* $D_1, \ldots, D_n$ to be

$$\begin{aligned} \gamma_{D_1,\ldots,D_n} : (D_1)_s \times \ldots \times (D_n)_s &\longrightarrow \mathbb{P}\left(\mathfrak{m}_0^n/\mathfrak{m}_0^{n+1}\right) \\ (x_1, \ldots, x_n) &\longmapsto \mathbb{C} \cdot j_0^n \left(\bigotimes_{i=1}^{n} t_{x_i}^* s_i \right) . \end{aligned}$$

We will need the following

Lemma 1.1 *If $D_1, \ldots, D_m$ are ample reduced divisors, then the image of $\gamma_{D_1,\ldots,D_n}$ is not contained in a hyperplane.*

Proof. The ampleness of D_i implies that the image of γ_{D_i} is not contained in a hyperplane in $\mathbb{P}\left(\mathfrak{m}_0/\mathfrak{m}_0^2\right)$ (see [LB, Proposition 4.4.1]). The assertion then follows from the commutative diagram

$$\begin{tikzcd} (D_1)_s \times \ldots \times (D_n)_s \arrow[rr, "\gamma_{D_1,\ldots,D_n}"] \arrow[dr, "\gamma_{D_1}\times\ldots\times\gamma_{D_n}"'] & & \mathbb{P}\left(\mathfrak{m}_0^m/\mathfrak{m}_0^{m+1}\right) \\ & \prod_1^n \mathbb{P}\left(\mathfrak{m}_0/\mathfrak{m}_0^2\right) \arrow[ur, "\mu"'] & \end{tikzcd}$$

where μ is induced by the product map. $\square$

2. The main result

Theorem 2.1 *Let A be an abelian variety and let $L_1, \ldots, L_{k+1}$ be ample line bundles on A, $k \geq 1$. Assume that L_{k+1} has no fixed components. Then $L = L_1 + \ldots + L_{k+1}$ is $k-$jet ample.*

Proof. Let $y_1, \ldots, y_r \in A$ and integers $k_1, \ldots, k_r > 0$ with $\sum k_i = k+1$ be given. We have to show that the restriction map

$$H^0(L) \longrightarrow H^0\left(L \otimes \mathcal{O}_A/\left(\mathfrak{m}_{y_1}^{k_1} \otimes \ldots \otimes \mathfrak{m}_{y_r}^{k_r}\right)\right)$$

is surjective.

First we assume that one of the integers, say k_1, satisfies $k_1 \geq 2$.

Claim 1. It is enough to show that the restriction map

$$H^0\left(L \otimes \widetilde{\mathfrak{m}} \otimes \mathfrak{m}_{y_1}^{k_1-1}\right) \longrightarrow H^0\left(L \otimes \mathfrak{m}_{y_1}^{k_1-1}/\mathfrak{m}_{y_1}^{k_1}\right) \qquad (*)$$

is surjective, where $\widetilde{\mathfrak{m}} := \bigotimes_{i=2}^{r} \mathfrak{m}_{y_i}^{k_i}$.

In fact, by induction and [BaSz, Theorem 1.1] we may assume that $H^0(L) \longrightarrow H^0\left(L \otimes \mathcal{O}_A / \left(\widetilde{\mathfrak{m}} \otimes \mathfrak{m}_{y_1}^{k_1-1}\right)\right)$ is surjective; so Claim 1 follows from the following exact diagram:

$$\begin{array}{ccccccc}
0 \to H^0\left(L \otimes \widetilde{\mathfrak{m}} \otimes \mathfrak{m}_{y_1}^{k_1-1}\right) \to & & H^0(L) & & \to H^0\left(L \otimes \mathcal{O}_A / \left(\widetilde{\mathfrak{m}} \otimes \mathfrak{m}_{y_1}^{k_1-1}\right)\right) \to 0 \\
\downarrow & & \downarrow & & \| \\
0 \to H^0\left(L \otimes \mathfrak{m}_{y_1}^{k_1-1} / \mathfrak{m}_{y_1}^{k_1}\right) & \to & H^0\left(L \otimes \mathcal{O}_A / \left(\widetilde{\mathfrak{m}} \otimes \mathfrak{m}_{y_1}^{k_1}\right)\right) & \to & H^0\left(L \otimes \mathcal{O}_A / \left(\widetilde{\mathfrak{m}} \otimes \mathfrak{m}_{y_1}^{k_1-1}\right)\right) \to 0 \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

It remains to prove the surjectivity of $(*)$. Suppose the contrary. Then there is a hyperplane $H \subset \mathbb{P}\left(L \otimes \mathfrak{m}_{y_1}^{k_1-1} / \mathfrak{m}_{y_1}^{k_1}\right)$ such that for all sections $s \in H^0(L)$ the conditions

$$j_{y_i}^{k_i-1}(s) = 0 \text{ for } 2 \le i \le r \text{ and } j_{y_1}^{k_1-2}(s) = 0 \tag{1}$$

imply $\mathbb{C} \cdot j_{y_1}^{k_1-1}(s) \in H$. The idea now is to construct sections satisfying (1) and to use Lemma 1.1 to get a contradiction. It is convenient to renumber the bundles $L_1, \ldots, L_k$ by double subscripts in the following way:

$$L_{1,1}, \ldots, L_{1,k_1-1}, L_{2,1}, \ldots, L_{2,k_2}, \ldots, L_{r,1}, \ldots, L_{r,k_r} \ .$$

This is possible since $(k_1 - 1) + k_2 + \ldots + k_r = k$. Let Ω be the set of subscripts (i, l), i.e. $\Omega = \{(i,l) \mid 1 \le i \le r,\ 1 \le l \le k_i \text{ for } 2 \le i \le r \text{ and } 1 \le l \le k_i - 1 \text{ for } i = 1\}$. Now for every $(i, l) \in \Omega$ let $\Theta_{i,l} \in |L_{i,l}|$ be a reduced translation-free divisor. Such divisors exist according to [LB, Proposition 4.1.7 and Lemma 4.1.8], since all bundles $L_{i,l}$ are ample. For every $(i, l) \in \Omega$ with $i \ge 2$ we choose a point

$$x_{i,l} \in t_{y_i}^* \Theta_{i,l} \text{ such that } x_{i,l} \notin t_{y_1}^* \Theta_{i,l} \ . \tag{2}$$

This is possible, since otherwise we would have $t_{y_i}^* \Theta_{i,l} = t_{y_1}^* \Theta_{i,l}$ implying a contradiction with $y_1 \ne y_i$ for $i \ne 1$.

Let $s_{1,l} \in H^0(L_{1,l})$ be a section defining $\Theta_{1,l}$ for $l = 1, \ldots, k_1 - 1$. Then for any choice of points $x_{1,l} \in t_{y_1}^* \Theta_{1,l}$ the section

$$s_1 := t_{x_{1,1}}^* s_{1,1} \otimes \ldots \otimes t_{x_{1,k_1-1}}^* s_{1,k_1-1}$$

satisfies $j_{y_1}^{k_1-2}(s_1) = 0$.

Claim 2. There is a nowhere dense subset S of $t_{y_1}^* \Theta_{1,1}$ such that for all $x_{1,1} \in t_{y_1}^* \Theta_{1,1} \setminus S$ the following condition holds: there is a divisor $\Theta_{k+1} \in |L_{k+1}|$ and a point x_{k+1} such that $y_1 \notin t_{x_{k+1}}^* \Theta_{k+1}$ and

$$t_{x_{1,1}}^* \Theta_{1,1} + \ldots + t_{x_{r,k_r}}^* \Theta_{r,k_r} + t_{x_{k+1}}^* \Theta_{k+1} \in |L|.$$

Proof of Claim 2. Consider the homomorphism

$$\begin{aligned} \phi : A \times A &\longrightarrow \operatorname{Pic}^0(A) \\ (a_1, a_2) &\longmapsto t^*_{x_{1,2}} L_{1,2} - L_{1,2} + \ldots + t^*_{x_{r,k_r}} L_{r,k_r} - L_{r,k_r} \\ &\qquad + t^*_{a_1} L_{1,1} - L_{1,1} + t^*_{a_2} L_{k+1} - L_{k+1} \ . \end{aligned}$$

Let π_1, π_2 be the projections of the kernel of ϕ onto the first resp. the second factor. They are surjective and finite, because $L_{1,1}$ and L_{k+1} are ample (compare also [BaSz, Proof of Theorem 1.1]).

Suppose now that the assertion of Claim 2 is false. This means that there is an open subset $D \subset t^*_{y_1}\Theta_{1,1}$ such that for all $x_{1,1} \in D$ and all $x_{k+1} \in \pi_2\pi_1^{-1}(x_{1,1})$ the point y_1 is a base point of $t^*_{x_{k+1}} L_{k+1}$ i.e. $y_1 \in t^*_{x_{k+1}}\Theta$ for all $\Theta \in |L_{k+1}|$, or equivalently $x_{k+1} \in t^*_{y_1}\Theta$ for all $\Theta \in |L_{k+1}|$. It follows that $\pi_2\pi_1^{-1}(D) \subset t^*_{y_1}\Theta$ for all $\Theta \in |L_{k+1}|$. But this means that $t^*_{y_1}L_{k+1}$ has a fixed component, a contradiction. This proves Claim 2.

Now let $x_{1,1} \in t^*_{y_1}\Theta_{1,1} \setminus S$ and let x_{k+1} and Θ_{k+1} be chosen as in Claim 2. Further, let s_2 be a section defining the divisor

$$t^*_{x_{2,1}}\Theta_{1,2} + \ldots + t^*_{x_{r,k_r}}\Theta_{r,k_r} + t^*_{x_{k+1}}\Theta_{k+1} \ .$$

Then $s := s_1 \otimes s_2 \in H^0(L)$ satisfies conditions (1). Therefore we conclude that $\mathbb{C} \cdot j^{k_1-1}_{y_1}(s) \in H$. Since $s_2(y_1) \neq 0$ it follows that $\mathbb{C} \cdot j^{k_1-1}_{y_1}(s_1) \in H'$, where H' is the image of H in $\mathbb{P}\left(\mathfrak{m}_0^{k_1-1}/\mathfrak{m}_0^{k_1}\right)$ via the canonical isomorphism. Since this holds for arbitrary points $x_{1,2}, \ldots, x_{1,k_1-1}$ of $t^*_{y_1}\Theta_{1,2}, \ldots, t^*_{y_1}\Theta_{1,k_1-1}$ and all $x_{1,1} \in t^*_{y_1}\Theta_{1,1} \setminus S$, we thus have shown that the image of the restriction of the map

$$\begin{aligned} \prod_{l=1}^{k_1-1} \left(t^*_{y_1}\Theta_{1,l}\right)_s &\longrightarrow \mathbb{P}\left(\mathfrak{m}_0^{k_1-1}/\mathfrak{m}_0^{k_1}\right) \\ (x_{1,1}, \ldots, x_{1,k_1-1}) &\longmapsto \mathbb{C} \cdot j^{k_1-1}_{y_1}(s_1) \end{aligned}$$

to a dense subset is contained in a hyperplane. But then the image of the map itself is contained in this hyperplane, a contradiction with Lemma 1.1.

It remains to deal with the case $k_1 = \ldots = k_{k+1} = 1$. By symmetry and by Claim 1 it is enough to show that there is a section $s \in H^0(L)$ vanishing at $y_1, \ldots, y_k$ and not vanishing at y_{k+1}. Such a section may be constructed directly as follows. Let $\Theta_1, \ldots, \Theta_k$ be reduced translation-free divisors in $|L_1|, \ldots, |L_k|$ respectively. For $1 \leq i \leq k$ there are points $x_i \in t^*_{y_i}\Theta_i \setminus t^*_{y_{k+1}}\Theta_i$. This means that $y_i \in t_{x_i}\Theta_i$ and $y_{k+1} \notin t_{x_i}\Theta_i$. According to [LB, Lemma 4.1.8 and Theorem 4.3.5] there is a reduced, irreducible translation-free divisor $\Theta_{k+1} \in |L_{k+1}|$. Exactly as in Claim 2 we can choose the point x_k in such a way that there is a point $x_{k+1} \in A$ such that $y_{k+1} \notin t^*_{x_{k+1}}\Theta_{k+1}$ and

$$t^*_{x_1}\Theta_1 + \ldots + t^*_{x_{k+1}}\Theta_{k+1} \in |L|.$$

Evidently a section $s \in H^0(L)$ defining the above divisor satisfies all the requirements. This completes the proof of the theorem. $\square$

Corollary 2.2 *Let A be an abelian variety and let $L_1, \ldots, L_{k+2}$ be ample line bundles on A, $k \geq 0$. Then $L_1 + \ldots + L_{k+2}$ is $k-$jet ample.*

Proof. This follows immediately from Theorem 2.1 because $L'_{k+1} := L_{k+1} + L_{k+2}$ is globally generated ([BaSz, Theorem 1.1a]). □

In particular, we have

Corollary 2.3 *Let A be an abelian variety and let L be an ample line bundle on A of type $(d_1, \ldots, d_g)$. If $d_1 \geq k+2$, then L is k-jet ample.*

Now we show that in general a tensor product of only $k+1$ ample line bundles on an abelian variety is not k-jet ample, even that it is not k-very ample or k-spanned.

Proposition 2.4 *Let $E_1, \ldots, E_g$ be elliptic curves, $g \geq 1$, and let $A = E_1 \times \ldots \times E_g$ with the canonical principal polarization*

$$L = \mathcal{O}_A \left(\sum_{i=1}^{g} E_1 \times \ldots \times E_{i-1} \times \{0\} \times E_{i+1} \times \ldots \times E_g \right) .$$

Then for any $k \geq 0$ the line bundle $(k+1)L$ is not k-spanned.

Proof. Consider the elliptic curve $E = E_1 \times \{0\} \times \ldots \times \{0\}$ on A. It is enough to show:

($*$) The restricted bundle $(k+1)L|_E$ is not k-very ample.

For this note that the notions of k-very ampleness and k-spannedness coincide on curves.

To prove ($*$), we can invoke Proposition 2.1 of [BeSo3] which states that for a k-very ample line bundle M on a curve C one always has $h^0(M) \geq k+1$ with equality only in case C is a smooth rational curve.

As for another way to verify ($*$), it is easy to see that one can choose $k+1$ points on E such that any divisor in the system $|(k+1)L|_E|$, which contains k of these points, also contains the remaining point because of Abel's theorem. □

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