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Smooth Kummer surfaces in projective three-space

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Abstract

In this note we prove the existence of smooth Kummer surfaces in projective three-space containing sixteen mutually disjoint smooth rational curves of any given degree.

Introduction

Let X be a smooth quartic surface in projective three-space $\mathbb{P}^3$. As a consequence of Nikulin's theorem [6] X is a Kummer surface if and only if it contains sixteen mutually disjoint smooth rational curves. The classical examples of smooth Kummer surfaces in $\mathbb{P}^3$ are due to Traynard (see [8] and [4]). They were rediscovered by Barth and Nieto [2] and independently by Naruki [5]. These quartic surfaces contain sixteen skew lines. In [1] it was shown by different methods that there also exist smooth quartic surfaces in $\mathbb{P}^3$ containing sixteen mutually disjoint smooth *conics*.

Motivated by these results it is then natural to ask if, for any given integer $d \geq 1$, there exist smooth quartic surfaces in $\mathbb{P}^3$ containing sixteen mutually disjoint smooth rational curves of degree d. The aim of this note is to show that the method of [1] can be generalized to answer this question in the affirmative. We show:

Theorem. *For any integer $d \geq 1$ there is a three-dimensional family of smooth quartic surfaces in $\mathbb{P}^3$ containing sixteen mutually disjoint smooth rational curves of degree d.*

We work throughout over the field $\mathbb{C}$ of complex numbers.

1. Preliminaries

Let (A, L) be a polarized abelian surface of type $(1, 2d^2 + 1)$, $d \geq 1$, and let L be symmetric. Denote by $e_1, \ldots, e_{16}$ the halfperiods of A. We are going to consider the

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non-complete linear system

$$\left| \mathcal{O}_A(2L) \otimes \bigotimes_{i=1}^{16} \mathfrak{m}_{e_i}^d \right|^{\pm} \qquad (*)$$

of even respectively odd sections of $\mathcal{O}_A(2L)$ vanishing in $e_1, \ldots, e_{16}$ to the order d. (As for the sign $\pm$ we will always use the following convention: we take $+$ if d is even, and $-$ if d is odd.) A parameter count shows that the expected dimension of this linear system is 4. In fact, we will show that it yields an embedding of the smooth Kummer surface X of A into $\mathbb{P}_3$ in the generic case. The linear system $(*)$ corresponds to a line bundle M_L on X such that

$$\pi^* M_L = \mathcal{O}_{\widetilde{A}}\left(2\sigma^* L - d \sum_{i=1}^{16} E_i\right) H^0(X, M_L) \cong H^0\left(A, \mathcal{O}_A(2L) \otimes \bigotimes_{i=1}^{16} \mathfrak{m}_{e_i}^d\right)^{\pm} .$$

Here $\sigma : \widetilde{A} \longrightarrow A$ is the blow-up of A in the halfperiods, $E_1, \ldots, E_{16} \subset \widetilde{A}$ are the exceptional curves and $\pi : \widetilde{A} \longrightarrow X$ is the canonical projection. The images of $E_1, \ldots, E_{16}$ under π will be denoted by $D_1, \ldots, D_{16}$.

We will need the following lemma:

Lemma 1.1 *Let the surfaces A and X and the line bundles L and M_L be as above. Further, let $C \subset X$ be an irreducible curve, different from $D_1, \ldots, D_{16}$, and let $F = \sigma_* \pi^* C$ be the corresponding symmetric curve on A. Then*

(a) $M_L^2 = 4$ *and* $M_L D_i = d$ *for* $1 \le i \le 16$,

(b) $F^2 = 2C^2 + \sum_{i=1}^{16} \operatorname{mult}_{e_i}(F)^2$, *and*

(c) $LF = M_L C + \frac{d}{2} \sum_{i=1}^{16} \operatorname{mult}_{e_i}(F)$.

The proof consists in an obvious calculation.

2. Bounding degrees and multiplicities

Here we show two technical statements on the degrees and multiplicities of symmetric curves. We start with a lemma which bounds the degree of a symmetric curve on A in terms of the degree of the corresponding curve on the smooth Kummer surface of A:

Lemma 2.1 *Let $C \subset X$ be an irreducible curve, different from $D_1, \ldots, D_{16}$, and let $F = \sigma_* \pi^* C$.*

(a) *If* $M_L C = 0$*, then* $LF \le 2\left(1 - C^2\right) d^2 + 16$.

(b) *If* $M_L C > 0$*, then* $LF \le 4\left(M_L C - \dfrac{C^2}{M_L C}\right) d^2 + 9 M_L C$.

Proof. For $\gamma \ge 0$ apply Hodge index to the line bundle M_L and the divisor $C + \frac{\gamma}{d} D_i$:

$$M_L^2 \left(C + \frac{\gamma}{d} D_i\right)^2 \le \left(M_L C + \frac{\gamma}{d} M_L D_i\right)^2 .$$

Using Lemma 1.1(a) and the equality $CD_i = \mathrm{mult}_{e_i}(F)$ we get

$$\mathrm{mult}_{e_i}(F) \le \left(\frac{(M_L C)^2}{8\gamma} + \frac{\gamma}{8} + \frac{M_L C}{4} - \frac{C^2}{2\gamma}\right) d + \frac{\gamma}{d} ,$$

hence by Lemma 1.1(c)

$$LF \le \left(\frac{(M_L C)^2}{\gamma} + \gamma + 2 M_L C - \frac{4C^2}{\gamma}\right) d^2 + M_L C + 8\gamma .$$

Now the assertion follows by setting $\gamma = 2$ in case $M_L C = 0$ and by setting $\gamma = M_L C$ otherwise. □

Further, we will need the following inequality on multiplicities of symmetric curves:

Lemma 2.2 *Let* $F \subset A$ *be a symmetric curve such that* $\mathcal{O}_A(F)$ *is of type* $(1, e)$ *with* e *odd. Then*

$$\sum_{i=1}^{16} \mathrm{mult}_{e_i}(F)^2 \ge \frac{1}{16}\left(\sum_{i=1}^{16} \mathrm{mult}_{e_i}(F)\right)^2 + \frac{15}{4} .$$

Proof. For $k \ge 0$ define the integers n_k by

$$n_k =_{\mathrm{def}} \#\{i \mid m_i = k,\ 1 \le i \le 16\} .$$

Abbreviating $m_i = \mathrm{mult}_{e_i}(F)$ we then have

$$\sum_{i=1}^{16} m_i = \sum_{k \ge 0} k n_k \sum_{i=1}^{16} m_i^2 = \sum_{k \ge 0} k^2 n_k .$$

The polarized abelian surface $(A, \mathcal{O}_A(F))$ is the pull-back of a principally polarized abelian surface (B, P) via an isogeny $\varphi : A \longrightarrow B$ of odd degree. The Theta divisor $\Theta \in |P|$ passes through six halfperiods with multiplicity one and through ten halfperiods with even multiplicity. Therefore the symmetric divisor $F \in |\varphi^* P|$

is of odd multiplicity in six halfperiods and of even multiplicity in ten halfperiods or vice versa. So we have

$$\sum_{k\equiv 0(2)} n_k = a \sum_{k\equiv 1(2)} n_k = b \ , \tag{1}$$

where $(a,b)=(6,10)$ or $(a,b)=(10,6)$.

Under the restriction (1) the difference

$$\sum k^2 n_k - \frac{1}{16}\left(\sum k n_k\right)^2$$

is minimal, if for some integer $k_0 \geq 0$ we have

$$n_{k_0}=10,\ n_{k_0+1}=6 \text{ or } n_{k_0}=6,\ n_{k_0+1}=10 \ .$$

In this case we get $\sum k^2 n_k - \frac{1}{16}\left(\sum k n_k\right)^2 = \frac{15}{4}$, which implies the assertion of the lemma. □

3. Kummer surfaces with sixteen skew rational curves of given degree

The aim of this section is to show:

Theorem 3.1 *Let (A,L) be a polarized abelian surface of type $(1,2d^2+1)$, $d\geq 1$. Assume $\rho(A)=1$. Then the map $\varphi_{M_L}: X \longrightarrow \mathbb{P}^3$ defined by the linear system $|M_L|$ is an embedding. The image surface $\varphi_{M_L}(X)$ is a smooth quartic surface containing sixteen mutually disjoint smooth rational curves of degree d.*

In particular, this implies the theorem stated in the introduction.

Proof. Using Riemann-Roch, Kodaira vanishing and Lemma 1.1(a), we will be done as soon as we can show that M_L is very ample. For $d=1$ this follows from [3], whereas for $d=2$ it follows from [1]. So we may assume $d\geq 3$ in the sequel.

(a) First we show that M_L is globally generated. A possible base part B of the system $\left|\mathcal{O}_A(2L)\otimes\bigotimes_{i=1}^{16}\mathfrak{m}_{e_i}^d\right|^{\pm}$ is totally symmetric, so B is algebraically equivalent to some even multiple of L, which is impossible for dimensional reasons. It remains the possibility that one – hence all – of the curves D_i is fixed in $|M_L|$. So $M_L-\mu\sum D_i$ is free for some $\mu\geq 1$. But $\left(M_L-\mu\sum D_i\right)^2 = 4-32\mu d-32\mu^2<0$, a contradiction.

(b) Our next claim is that M_L is ample. Otherwise there is an irreducible (-2)-curve $C\subset X$ such that $M_L C=0$. Lemma 1.1 shows that we have

$$LF=\frac{d}{2}\sum m_i F^2 = -4+\sum m_i^2$$

for the symmetric curve $F=\sigma_*\pi^*C$ with multiplicities $m_i=\operatorname{mult}_{e_i}(F)$. According to Lemma 2.1 the degree of F is bounded by

$$LF\leq 6d^2+16 \ . \tag{2}$$

Since L is a primitive line bundle, the assumption on the Néron-Severi group of A implies that $\mathcal{O}_A(F)$ is algebraically equivalent to some multiple pL, $p \geq 1$, thus we have $LF = pL^2 = p(4d^2+2)$ and then (2) implies $p = 1$ because of our assumption $d \geq 3$. So we find

$$8d^2 + 4 = 2LF = d \sum m_i$$

and reduction mod d shows that necessarily $d = 4$. But in this case $\sum m_i$ would be odd, which is impossible (cf. [3]).

(c) Finally we prove that M_L is very ample. Suppose the contrary. Saint-Donat's criterion [7, Theorem 5.2 and Theorem 6.1(iii)] then implies the existence of an irreducible curve $C \subset X$ with $M_L C = 2$ and $C^2 = 0$. So we have

$$LF = 2 + \frac{d}{2} \sum m_i F^2 = \sum m_i^2$$

for the corresponding symmetric curve $F = \sigma_* \pi^* C$. Lemma 2.1 yields the estimate

$$LF \leq 8d^2 + 18 \ .$$

As above $\mathcal{O}_A(F)$ is algebraically equivalent to some multiple pL, $p \geq 1$, hence we get

$$p\left(4d^2+2\right) = pL^2 \leq 8d^2 + 18 \ ,$$

which implies $p \leq 2$. If we had $p = 2$ then reduction mod d of the equation

$$2\left(4d^2+2\right) = 2 + \frac{d}{2} \sum m_i$$

would give $d = 4$. But in this case we have $\sum m_i = 65$, which is impossible.

So the only remaining possibility is $p = 1$, thus

$$4d^2 + 2 = 2 + \frac{d}{2} \sum m_i = \sum m_i^2 \ .$$

But a numerical check shows that this contradicts Lemma 2.2. This completes the proof of the theorem. □

Remark 3.2 We conclude with a remark on the genericity assumption on the abelian surface A. It is certainly not true that the line bundle M_L is very ample for *every* polarized abelian surface (A, L) of type $(1, 2d^2+1)$. Consider for instance the case where $A = E_1 \times E_2$ is a product of elliptic curves and $L = \mathcal{O}_A(\{0\} \times E_2 + (2d^2+1)E_1 \times \{0\})$. Here, taking $C \subset X$ to be curve corresponding to $E_1 \times \{0\}$, we have

$$M_L C = 1 - 2d < 0 \ ,$$

so in this case M_L is not even ample or globally generated.

References

[1] Barth, W., Bauer, Th.: Smooth quartic surfaces with 352 conics. Manuscripta math. 85, 409-417 (1994)

[2] Barth, W., Nieto, I.: Abelian surfaces of type $(1,3)$ and quartic surfaces with 16 skew lines. J. Algebraic Geometry 3, 173-222 (1994)

[3] Bauer, Th.: Projective images of Kummer surfaces. Math. Ann. 299, 155-170 (1994)

[4] Godeaux, L.: Sur la surface du quatrième ordre contenant trente-deux droites. Académie Royale de Belgique, Bulletin de la Classe des Sciences, 5. sér., 25, 539-552 (1939)

[5] Naruki, I.: On smooth quartic embedding of Kummer surfaces. Proc. Japan Acad. 67, Ser. A, 223-225 (1991)

[6] Nikulin, V.V.: On Kummer surfaces. Math. USSR Izvestija, Vol.9, No.2, 261-275 (1975)

[7] Saint-Donat, B.: Projective models of K3 surfaces. Amer. J. Math. 96, 602-639 (1974)

[8] Traynard, E.: Sur les fonctions thêta de deux variables et les surfaces hyperelliptiques. Ann. scient. Éc. Norm. Sup., 3. sér., t. 24, 77-177 (1907)

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