Building and measuring mathematical sophistication in pre-service mathematics teachers

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We advocate that fostering mathematical sophistication should be a main role that advanced mathematics contents courses play in the university education of pre-service teachers.

Mathematical sophistication – a desired outcome of advanced mathematics courses

University mathematics teacher education programs face a fundamental problem of what Felix Klein (1924) called the Doppelte Diskontinuität (double discontinuity). The first discontinuity occurs with the transition from school to university mathematics, and the second discontinuity concerns whether this university education has the desired impact on their future work as mathematics teachers. (See Hefendehl-Hebeker (2013) for an overview of the problem and of contemporary efforts to tackle it). At issue with the second discontinuity is whether pre-service teachers are provided opportunities in their university coursework to learn the mathematics content knowledge (MCK), mathematics pedagogical content knowledge (PCK), and pedagogical knowledge (Shulman 1986) required for the work of teaching. Within the domain of PCK, Bass and Ball (2004) have identified and developed instruments to measure what they have termed *Mathematical Knowledge for Teaching*, which includes knowing which concepts best support students' understanding, and recognizing the nature of students' various conceptions and misconceptions. This knowledge is specific to the content of school mathematics, and is likely not to be fostered directly through advanced mathematics coursework. Therefore, an important issue within the domain of MCK is the role that advanced mathematics coursework has in developing the mathematics content knowledge that teachers actually need to teach school mathematics.

Szydlik and Seaman (2007) have identified specific aspects of MCK that are not content-specific, but rather knowledge of how to do mathematics, when they proposed the construct of *Mathematical Sophistication*. This construct refers to a person's mathematical behavior – the avenues of doing mathematics that one has at their disposal – and consists of an internalization of the values, behaviors, and habits of mind of the mathematical community that are powerful in learning new mathematics. The concept is rooted in a sociocultural perspective on mathematics learning (Bauersfeld 1979; Resnick 1989; Schoenfeld 1992): Through a process of enculturation in what it means to do mathematics, the learner aquires a mathematical point of view – thus "seeing the world in ways like mathematicians do" (Schoenfeld 1992).

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The development of the mathematical sophistication concept (along with a framework of norms characterizing it) by Seaman and Szydlik (2007) was motivated by their study in which a majority of pre-service elementary teachers were unable to use a teacher resource to make sense of an unfamiliar mathematical concept – a failure that the authors attributed to the participants not being able to think and act like mathematicians would have. This suggests that the pedagogically powerful forms of mathematical content knowledge intersect to a considerable extent with the forms of knowledge that allow mathematicians to create new mathematics. We agree with Seaman and Szydlik that building mathematical sophistication is critical not only for prospective research mathematicians, but for anyone engaged in mathematical learning – it should therefore be the main role that mathematics coursework plays in the preparation of teachers. While the mathematics content of advanced university mathematics courses might not have an obvious counterpart in school mathematics, an explicit goal of this coursework should be to allow students to acquire traits of mathematical behavior that empowers them to do and make sense of mathematics, and be able to enculturate these behaviors in their own future classrooms.

Building mathematical sophistication

Motivating students to value mathematical sophistication. A survey carried out by the first author (unpublished) indicates that a large portion of pre-service teachers is interested in university mathematics only as far as it is visibly related to their future jobs as teachers, rather than as an interesting scientific endeavor in and of itself. It is therefore important to convince students that mathematical sophistication is in fact useful – and in many situations even a requirement – for successful teaching. One approach in this direction are *interface activities* („Schnittstellenaktivitäten") (Bauer und Partheil 2009; Bauer 2013a,b), which consist of specific homework problems ("Schnittstellenaufgaben") discussed in special recitation sections ("Schnittstellenübungen"), designed to establish connections between school mathematics and university mathematics – such as problems that highlight the use of advanced techniques from university mathematics in order to gain deeper insight into topics appearing in school mathematics (category C in Bauer 2013a and Bauer 2013b).

Designing advanced courses that help students gain mathematical sophistication. Mathematical behavior is a facet of mathematics knowledge that is rarely made explicit in mathematics content courses – perhaps it is often assumed that students will notice them implicitly. However, we argue:

(1) Mathematics content courses should make mathematical behavior more explicit.
(2) Mathematics content courses should involve students in more activities that require authentic mathematical behavior.

Here (1) entails showing avenues of knowing that the mathematical community has developed in general, but also "disclosing" the mental models and strategies used by the educator concerning the currently studied concepts and problems, respectively, in order to foster cognitive apprenticeship (Collins et al. 1989). The lecturing tradition in mathematics so far does not put much emphasis on these aspects – the focus is predominantly on the finished products (expressed in definitions, theorems and proofs) rather than on the acting mathematician's behavior. As for (2), reactions on the part of university educators might vary in a wide spectrum between the statements “We do this anyway” and “This is too difficult for the average student”. While it is true that challenging “Prove that …” problems can involve a variety of mathematical activities, it should be noticed that they do not cover the whole spectrum of mathematical behavior as conceptualized by the list of traits of mathematical sophistication from Seaman and Szydlik (2007). We argue that such activities are very well possible at every stage of mathematical education. (See also Bauer 2013c, where a case is made that this applies to school mathematics as well.) The example below, which involves the activities of *conjecturing* and *defining*, is given to support this point of view. We agree with Belnap and Parrot (2013) that conjecturing is a valuable mathematical activity for students, as it appears to involve many of the traits of mathematical behavior that Seaman and Szydlik as well as Schoenfeld (1992) identified. The example (see the box) shows an exercise problem from a course of the first author on Elementary Algebraic Geometry, which encourages experimenting with examples, verbalizing expectations, as well as stating and proving conjectures. Compare it to a version of type “Prove that for every curves of degree d, the intersection with a line …” – the same theorem is being proved, but the mathematical activities differ substantially.

Exercise: Intersection of plane algebraic curves with lines

(a) Find all lines that intersect the curve

$$\{(x, y) \in \mathbb{C}^2 \mid y^2 = x^3\}$$

in 1 point, 2 points, 3 points. (A picture of the curve might be useful to get some intuition.)

(b) Why can't there be more than 3 points of intersection?

(c) Suppose a curve $C \subset \mathbb{C}^2$ is given by a polynomial of degree d. Based on the previous example, what possibilities do you expect for the number of intersection points of C with lines? State a conjecture and prove it.

Measuring mathematical sophistication

Szydlik, Kuennen and Seaman (2009) developed a 25-item multiple-choice instrument that attempts to measure a student's level of mathematical sophistication with items designed for the following traits: 1) find and understand patterns, 2) classify and characterize objects based on structure, 3) make and test conjectures, 4) create models of mathematical objects, 5) value precise definitions, 6) value an understanding of why relationships make sense, 7) value logical arguments as sources of conviction, 8) have fine distinctions about language, and 9) value symbolic representations and notation. We were interested in answering the following questions:

1. What kind of adaptations are necessary for use of the items with German students?
2. Is this instrument, which was designed for use with elementary and middle school preservice teachers, also meaningful when used with pre-service Gymnasium teachers?
3. In which ways do beginning students show different mathematical sophistication than ending students (novice-expert comparison)?

As for 1), we found that few adaptations beyond mere language translation were necessary. (This should be seen in contrast with Delaney et al. 2008, where a number of changes accounting for cross-cultural differences were deemed necessary.) Preliminary results for 2) suggest that the items work well with pre-service Gymnasium teachers. This might be explained by the fact that the items are by design not bound to specific mathematical content, as they aim at measuring behavior that results from coursework rather than content that occurs in coursework. As for 3) we found in a subset of the items a significant difference between novices and experts (i.e., beginning and ending students), while little difference for a second subset. Further research is necessary in order to explain these findings – in particular it would be extremely interesting to uncover which of the findings can be attributed to the different nature of the chosen items (e.g. level of difficulty), and which to differing impact of university education on specific facets of mathematical sophistication.

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