Some Notes on Importance Sampling of a Hemisphere

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The following notes are intended to serve as my own reference and are not written as an easy-to-follow introduction.

1 Importance Sampling

Using the theory of importance sampling, we can approximate any integral by a sum:

$$\int_a^b \mathrm{f}(x)\,dx \approx \frac{1}{N}\sum_{n=1}^{N}\frac{\mathrm{f}(x_n)}{\mathrm{p}(x_n)} \quad , \tag{1}$$

where $p(x)$ is some arbitrary probability density function (PDF) that must fulfill the condition:

$$\int_a^b \mathrm{p}(x)\,dx = 1 \quad . \tag{2}$$

In theory, the best PDF (with the smallest variance) would be

$$\mathrm{p}(x) = \frac{\mathrm{f}(x)}{\int_a^b \mathrm{f}(x)} \quad , \tag{3}$$

which means the PDF should follow the shape of the function (i.e., the sampling density should be higher if the function values are higher).

2 Sampling of a Hemisphere

Let angle θ be the polar angle and ϕ the azimuthal angle of a spherical coordinate system. Futhermore, we have a function $\mathrm{s}(\theta, \phi)$ that evaluates a sample at a position (θ, ϕ) on the hemisphere. We like to solve the integral of the function $\mathrm{s}(\theta, \phi)$ over the hemisphere:

$$\int\limits_{\mathbf{\Omega}^+} \mathrm{s}(\theta, \phi)\; d\omega = \int\limits_{\phi=0}^{2\pi}\int\limits_{\theta=0}^{\frac{\pi}{2}} s(\theta, \phi)\; \sin(\theta)\, d\theta\, d\phi \tag{4}$$

In order to solve the integral with importance sampling, as a first very simple example, we could choose the PDF to be a uniform distribution:

$$\mathrm{p}(\theta,\phi) = \frac{1}{2\pi}\frac{1}{\pi/2} \quad . \tag{5}$$

The condition in Eq. (2) would be fulfilled because

$$\int\limits_{\phi=0}^{2\pi}\int\limits_{\theta=0}^{\frac{\pi}{2}} \frac{1}{2\pi}\frac{1}{\pi/2}\, d\theta\, d\phi = 1 \quad . \tag{6}$$

Now using Eq. (1) we get:

$$\begin{aligned}
&\int\limits_{\phi=0}^{2\pi}\int\limits_{\theta=0}^{\frac{\pi}{2}} \underbrace{\mathrm{s}(\theta,\phi)\,\sin(\theta)}_{\mathrm{f}(\theta,\phi)}\, d\theta\, d\phi && (7)\\
\approx&\frac{1}{J\,K}\sum_{k=1}^{K}\sum_{j=1}^{J}\frac{\mathrm{f}(\theta_j,\phi_k)}{\mathrm{p}(\theta_j,\phi_k)} && (8)\\
=&\frac{1}{J\,K}\sum_{k=1}^{K}\sum_{j=1}^{J}\frac{\mathrm{s}(\theta_j,\phi_k)\,\sin(\theta_j)}{\frac{1}{2\pi}\frac{1}{\pi/2}} && (9)\\
=&\sum_{1}^{K}\sum_{1}^{J}\mathrm{s}(\theta_j,\phi_k)\,\sin(\theta_j)\underbrace{\frac{\pi/2}{J}}_{\Delta\theta}\underbrace{\frac{2\pi}{K}}_{\Delta\phi} && (10)\\
=&\sum_{1}^{K}\sum_{1}^{J}\mathrm{s}(\theta_j,\phi_k)\,\sin(\theta_j)\Delta\theta\,\Delta\phi && (11)
\end{aligned}$$

Let the two random variables u and v be in range [0.0, 1.0] and have a uniform distribution. Then, for this choice of the PDF, the mapping from the two uniform random variables u and v to θ and ϕ is a simple linear relation:

$$\phi = 2\pi\, u \tag{12}$$

$$\theta = \frac{\pi}{2}\, v \tag{13}$$

This uniform sampling in θ and ϕ directions provides a solution to the integral. In fact, it is the same solution the Riemann sum would provide. However, this solution can be improved because the multiplication with the $\sin(\theta)$ function makes the contributions of samples very small when θ approaches zero. Therefore, let us choose a more suitable PDF:

$$\mathrm{p}(\theta,\phi) = c\,\sin(\theta) \quad , \tag{14}$$

where c is a constant that we can determine by evaluating the condition in Eq. (2):

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} c \sin(\theta)\, d\theta\, d\phi = 1 \tag{15}$$

$$2\pi\, c \left[-\cos(x) \right]_0^{\frac{\pi}{2}} = 1 \tag{16}$$

$$c = \frac{1}{2\pi} \quad . \tag{17}$$

The resulting PDF is:

$$\mathrm{p}(\theta, \phi) = \frac{1}{2\pi} \sin(\theta) \quad . \tag{18}$$

Now, using Eq. (1), we get:

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} \underbrace{\mathrm{s}(\theta, \phi) \sin(\theta)}_{\mathrm{f}(\theta_j, \phi_k)}\, d\theta\, d\phi \tag{19}$$

$$\approx \frac{1}{J\,K} \sum_{k=1}^{K} \sum_{j=1}^{J} \frac{\mathrm{f}(\theta_j, \phi_k)}{\mathrm{p}(\theta, \phi)} \tag{20}$$

$$= \frac{2\pi}{J\,K} \sum_{k=1}^{K} \sum_{j=1}^{J} \frac{\mathrm{s}(\theta_j, \phi_k) \sin(\theta_j)}{\sin(\theta_j)} \tag{21}$$

$$= \frac{2\pi}{J\,K} \sum_{k=1}^{K} \sum_{j=1}^{J} \mathrm{s}(\theta_j, \phi_k) \tag{22}$$

This looks perfect. However, how do we generate samples that have a probability distribution function of $\mathrm{p}(\theta, \phi) = \frac{1}{2\pi} \sin(\theta)$? To this end, we follow exactly the approach described in the PBR book[1] by Pharr and Humphreys and use the inversion method[2] for random variables. First, we compute the marginal density function:

$$\mathrm{p}_\theta(\theta) = \int_{\phi=0}^{2\pi} \mathrm{p}(\theta, \phi)\, d\phi = \int_{\phi=0}^{2\pi} \frac{1}{2\pi} \sin(\theta)\, d\phi = \frac{1}{2\pi} \sin(\theta) \int_{\phi=0}^{2\pi} d\phi = \sin(\theta) \tag{23}$$

Next, the conditional density for ϕ is computed:

$$\mathrm{p}(\phi|\theta) = \frac{\mathrm{p}(\theta, \phi)}{\mathrm{p}_\theta(\theta)} = \frac{1}{2\pi} \tag{24}$$

[1] M. Pharr, G. Humphreys: Physically Based Rendering (2010), ch. 13.6.1, pp. 643ff
[2] https://en.wikipedia.org/wiki/Inverse_transform_sampling

For both distributions, we compute the cumulative distribution function (CDF):

$$\mathrm{P}(\phi|\theta) = \int_0^{\phi} \frac{1}{2\pi} d\phi' = \frac{1}{2\pi}\phi \tag{25}$$

$$\mathrm{P}(\theta) = \int_0^{\theta} \sin(\theta') d\theta' = \Big[-\cos(\theta') \Big]_0^{\theta} = 1 - \cos(\theta) \tag{26}$$

Finally, in a last step, we apply the inversion:

$$u = \mathrm{P}(\phi|\theta) = \frac{1}{2\pi}\phi \quad \Rightarrow \quad \phi = 2\pi\, u \tag{27}$$

$$v = \mathrm{P}(\theta) = 1 - \cos(\theta) \quad \Rightarrow \quad \theta = \arccos(1 - v) \tag{28}$$

Thus, a PDF of $\mathrm{p}(\theta, \phi) = \frac{1}{2\pi} \sin(\theta)$ can be produces from two uniform random variables x and y in range [0.0, 1.0] and the mapping

$$\phi = 2\pi\, u \tag{29}$$

$$\theta = \arccos(1 - v) \tag{30}$$

If the sample positions (θ_j, ϕ_k) are generated with this PDF, the integral over the hemisphere can be computed by the simple formula from Eq. (22).

3 Sampling of a Cosine-Weighted Hemisphere

Next, we like to solve the integral of the function $\mathrm{s}(\theta, \phi)$ over a cosine-weighted hemisphere:

$$\int\limits_{\mathbf{\Omega}^+} \mathrm{s}(\theta, \phi)\, \cos(\theta)\, d\omega = \int\limits_{\phi=0}^{2\pi} \int\limits_{\theta=0}^{\frac{\pi}{2}} s(\theta, \phi)\, \cos(\theta) \sin(\theta)\, d\theta\, d\phi \tag{31}$$

Therefore, a suitable PDF is:

$$\mathrm{p}(\theta, \phi) = c\, \cos(\theta)\, \sin(\theta) \quad . \tag{32}$$

Again, we determine the constant c with:

$$\int\limits_{\phi=0}^{2\pi} \int\limits_{\theta=0}^{\frac{\pi}{2}} c\, \cos(\theta)\, \sin(\theta)\, d\theta\, d\phi = 1 \tag{33}$$

$$2\pi\, c \left[\frac{1}{2} \sin^2(\theta) \right]_0^{\frac{\pi}{2}} = 1 \tag{34}$$

$$c = \frac{1}{\pi} \quad . \tag{35}$$

Thus, the PDF is

$$\mathrm{p}(\theta,\phi) = \frac{1}{\pi}\,\cos(\theta)\,\sin(\theta) \quad . \tag{36}$$

Marginal density function:

$$\mathrm{p}_\theta(\theta) = \int\limits_{\phi=0}^{2\pi} \mathrm{p}(\theta,\phi)\,d\phi = \int\limits_{\phi=0}^{2\pi} \frac{1}{\pi}\,\cos(\theta)\,\sin(\theta)\,d\phi = 2\,\cos(\theta)\,\sin(\theta) \tag{37}$$

Conditional density for ϕ:

$$\mathrm{p}(\phi|\theta) = \frac{\mathrm{p}(\theta,\phi)}{\mathrm{p}_\theta(\theta)} = \frac{1}{2\pi} \tag{38}$$

Cumulative distribution function (CDF):

$$\mathrm{P}(\phi|\theta) = \int_0^\phi \frac{1}{2\pi} d\phi' = \frac{1}{2\pi}\phi \tag{39}$$

$$\mathrm{P}(\theta) = \int_0^\theta 2\,\cos(\theta')\,\sin(\theta')d\theta' = 2\left[\frac{1}{2}\sin^2(\theta')\right]_0^\theta = \sin^2(\theta) \tag{40}$$

Inversion:

$$u = \mathrm{P}(\phi|\theta) = \frac{1}{2\pi}\phi \quad \Rightarrow \quad \phi = 2\pi\,u \tag{41}$$

$$v = \mathrm{P}(\theta) = \sin^2(\theta) \quad \Rightarrow \quad \theta = \arcsin(\sqrt{v}) \tag{42}$$

Thus, a PDF of $\mathrm{p}(\theta,\phi) = \frac{1}{\pi}\,\cos(\theta)\,\sin(\theta)$ can be produces from two uniform random variables x and y in range [0.0, 1.0] and the mapping

$$\phi = 2\pi\,u \tag{43}$$

$$\theta = \arcsin(\sqrt{v}) \quad . \tag{44}$$

If the sample positions (θ_j,ϕ_k) are generated with this PDF, the integral over the cosine-weighted hemisphere can be computed by

$$\int\limits_{\phi=0}^{2\pi}\int\limits_{\theta=0}^{\frac{\pi}{2}} \underbrace{\mathrm{s}(\theta,\phi)\,\cos(\theta)\sin(\theta)}_{\mathrm{f}(\theta,\phi)}\,d\theta\,d\phi \tag{45}$$

$$\approx \frac{1}{J\,K}\sum_{k=1}^{K}\sum_{j=1}^{J}\frac{\mathrm{f}(\theta_j,\phi_k)}{\mathrm{p}(\theta,\phi)} \tag{46}$$

$$= \frac{\pi}{J\,K}\sum_{k=1}^{K}\sum_{j=1}^{J}\mathrm{s}(\theta_j,\phi_k) \quad . \tag{47}$$

4 Sampling of the Phong Specular Lobe over a Hemisphere

Next, we like to sample the specular component of the Phong (or Blinn-Phong) BRDF. The Phong BRDF requires sampling a cosine lobe with specular exponent n around the reflected direction (or half-vector direction in case of Blinn-Phong). Thus, the PDF is:

$$\mathrm{p}(\theta, \phi) = c\, \cos(\theta)^n\, \sin(\theta) \quad . \tag{48}$$

We determine the constant c with:

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} c\, \cos^n(\theta)\, \sin(\theta)\, d\theta\, d\phi = 1 \tag{49}$$

$$2\pi\, c \left[-\frac{\cos^{n+1}(\theta)}{n+1} \right]_0^{\frac{\pi}{2}} = 1 \tag{50}$$

$$c = \frac{n+1}{2\pi} \quad . \tag{51}$$

Thus, the complete PDF is

$$\mathrm{p}(\theta, \phi) = \frac{n+1}{2\pi}\, \cos^n(\theta)\, \sin(\theta) \quad . \tag{52}$$

Marginal density function:

$$\mathrm{p}_\theta(\theta) = \int_{\phi=0}^{2\pi} \mathrm{p}(\theta, \phi)\, d\phi = \int_{\phi=0}^{2\pi} \frac{n+1}{2\pi}\, \cos^n(\theta)\, \sin(\theta)\, d\phi = (n+1)\, \cos^n(\theta)\, \sin(\theta) \tag{53}$$

Conditional density for ϕ:

$$\mathrm{p}(\phi|\theta) = \frac{\mathrm{p}(\theta, \phi)}{\mathrm{p}_\theta(\theta)} = \frac{1}{2\pi} \tag{54}$$

Cumulative distribution function (CDF):

$$\mathrm{P}(\phi|\theta) = \int_0^\phi \frac{1}{2\pi} d\phi' = \frac{1}{2\pi}\phi \tag{55}$$

$$\mathrm{P}(\theta) = \int_0^\theta (n+1)\, \cos^n(\theta')\, \sin(\theta') d\theta' \tag{56}$$

$$= (n+1) \left[-\frac{\cos^{n+1}(\theta')}{n+1} \right]_0^\theta = 1 - cos^{n+1}(\theta) \tag{57}$$

Inversion:

$$u = \mathrm{P}(\phi|\theta) = \frac{1}{2\pi}\phi \quad \Rightarrow \quad \phi = 2\pi\, u \tag{58}$$

$$v = \mathrm{P}(\theta) = 1 - cos^{n+1} \quad \Rightarrow \quad \theta = \arccos\left((1-v)^{\frac{1}{n+1}}\right) \tag{59}$$

5 Sampling of the Microfacet GGX Specular Lobe over a Hemisphere

Now, we sample the specular component of a Cook-Torrance Microfacet BRDF. In particular, we want to sample the GGX/Trowbridge-Reitz version made popular in physical-based rendering by Disney[3] and Epic Games[4] with a specular D of:

$$\mathrm{D}(\theta) = \frac{\alpha^2}{\pi\left(\cos^2(\theta)(\alpha^2-1)+1\right)^2} \tag{60}$$

with $\alpha = r^2$, where r is the perceived roughness. As suggested in the Disney course notes (Appendix B.1), the PDF is chosen as:

$$\mathrm{p}(\theta,\phi) = \mathrm{D}(\theta)\cos(\theta)\sin(\theta) \quad . \tag{61}$$

This PDF should be already normalized. Let's check:

$$\int\limits_{\phi=0}^{2\pi}\int\limits_{\theta=0}^{\frac{\pi}{2}} \mathrm{D}(\theta)\cos(\theta)\,\sin(\theta)\,d\theta\,d\phi = 1 \tag{62}$$

$$\int\limits_{\phi=0}^{2\pi}\int\limits_{\theta=0}^{\frac{\pi}{2}} \frac{\alpha^2\cos(\theta)\sin(\theta)}{\pi\left((\alpha^2-1)\cos^2(\theta)+1\right)^2}\,d\theta\,d\phi = 1 \tag{63}$$

$$2\pi\left[\frac{\alpha^2}{\pi\,(2\alpha^2-2)\left((\alpha^2-1)\cos^2(\theta)+1\right)}\right]_0^{\frac{\pi}{2}} = 1 \tag{64}$$

$$1 = 1 \quad . \tag{65}$$

Marginal density function:

$$\mathrm{p}_\theta(\theta) = \int\limits_{\phi=0}^{2\pi} \mathrm{p}(\theta,\phi)\,d\phi = 2\pi\,\mathrm{D}(\theta)\cos(\theta)\sin(\theta) \tag{66}$$

Conditional density for ϕ:

$$\mathrm{p}(\phi|\theta) = \frac{\mathrm{p}(\theta,\phi)}{\mathrm{p}_\theta(\theta)} = \frac{1}{2\pi} \tag{67}$$

[3]Brent Burley: Physically Based Shading at Disney, SIGGRAPH 2012 course notes
[4]Brian Karis: Real Shading in Unreal Engine 4, SIGGRAPH 2013 course notes

Cumulative distribution function (CDF):

$$\mathrm{P}(\phi|\theta) = \int_0^{\phi} \frac{1}{2\pi} d\phi' = \frac{1}{2\pi}\phi \tag{68}$$

$$\begin{aligned}
\mathrm{P}(\theta) &= \int_0^{\theta} 2\pi\, \mathrm{D}(\theta') \cos(\theta') \sin(\theta') d\theta' && (69)\\
&= \int_0^{\theta} \frac{2\alpha^2 \cos(\theta') \sin(\theta')}{\left(\cos^2(\theta')(\alpha^2-1)+1\right)^2}\, d\theta' && (70)\\
&= \left[\frac{\alpha^2}{(\alpha^2-1)\left(\cos^2(\theta')(\alpha^2-1)+1\right)}\right]_0^{\theta} && (71)\\
&= \frac{\alpha^2}{(\alpha^2-1)\left(\cos^2(\theta)(\alpha^2-1)+1\right)} - \frac{\alpha^2}{(\alpha^2-1)\left(\cos^2(0)(\alpha^2-1)+1\right)} && (72)\\
&= \frac{\alpha^2}{(\alpha^2-1)\left(\cos^2(\theta)(\alpha^2-1)+1\right)} - \frac{1}{(\alpha^2-1)} && (73)
\end{aligned}$$

Inversion for ϕ:

$$u = \mathrm{P}(\phi|\theta) = \frac{1}{2\pi}\phi \quad \Rightarrow \quad \phi = 2\pi\, u \tag{74}$$

Inversion for θ:

$$v = \mathrm{P}(\theta) = \frac{\alpha^2}{(\alpha^2-1)\left(\cos^2(\theta)(\alpha^2-1)+1\right)} - \frac{1}{(\alpha^2-1)} \tag{75}$$

$$\begin{aligned}
v + \frac{1}{(\alpha^2-1)} &= \frac{\alpha^2}{(\alpha^2-1)\left(\cos^2(\theta)(\alpha^2-1)+1\right)} && (76)\\
\frac{v(\alpha^2-1)}{\alpha^2} + \frac{1}{\alpha^2} &= \frac{1}{\left(\cos^2(\theta)(\alpha^2-1)+1\right)} && (77)\\
\frac{v(\alpha^2-1)+1}{\alpha^2} &= \frac{1}{\left(\cos^2(\theta)(\alpha^2-1)+1\right)} && (78)\\
\frac{\alpha^2}{v(\alpha^2-1)+1} &= \cos^2(\theta)(\alpha^2-1)+1 && (79)\\
\frac{\alpha^2}{v(\alpha^2-1)+1} - 1 &= \cos^2(\theta)(\alpha^2-1) && (80)\\
\frac{\alpha^2}{(\alpha^2-1)\left(v(\alpha^2-1)+1\right)} - \frac{1}{(\alpha^2-1)} &= \cos^2(\theta) && (81)
\end{aligned}$$

$$\cos^2(\theta) = \frac{\alpha^2 - \big(v(\alpha^2 - 1) + 1\big)}{(\alpha^2 - 1)\,(v(\alpha^2 - 1) + 1)} \tag{82}$$

$$\cos^2(\theta) = \frac{(\alpha^2 - 1) - v(\alpha^2 - 1)}{(\alpha^2 - 1)\,(v(\alpha^2 - 1) + 1)} \tag{83}$$

$$\cos^2(\theta) = \frac{1 - v}{v(\alpha^2 - 1) + 1} \tag{84}$$

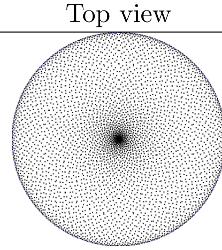
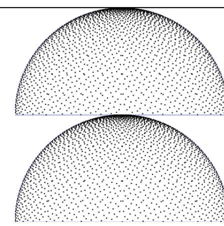
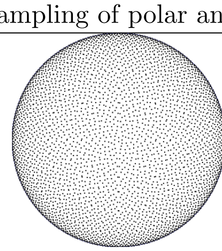
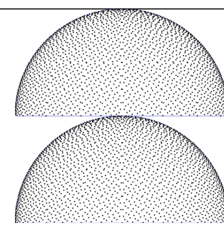
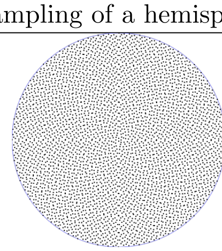
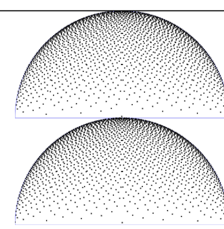
$$\cos(\theta) = \sqrt{\frac{1 - v}{v(\alpha^2 - 1) + 1}} \tag{85}$$

$$\theta = \arccos\left(\sqrt{\frac{1 - v}{v(\alpha^2 - 1) + 1}}\right) \tag{86}$$

6 Summary

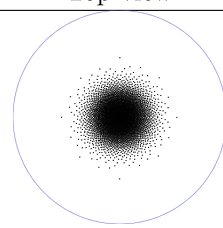
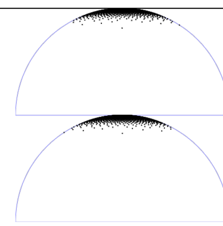
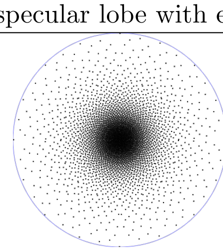
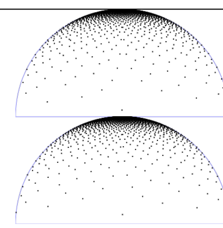
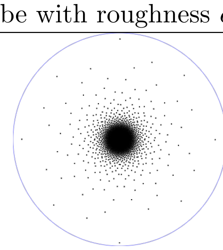
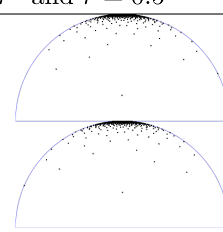
Tables 1 and 2 show the mappings from uniform random variables u and v in range [0.0, 1.0] to azimuthal angle ϕ and polar angle θ of a spherical coordinate system for sampling over a hemisphere. The shown visualizations of the sample distributions use the pseudo-random Hammersley sequence[5] and are created with this website: online demo

Table 1: Summary of mappings

PDF / Mapping	Top view	Front and back view
$p(\theta, \phi) = \frac{1}{2\pi} \frac{1}{\pi/2}$ $\phi = 2\pi\, u$ $\theta = \frac{\pi}{2}\, v$		
Uniform sampling of polar angles		
$p(\theta, \phi) = \frac{1}{2\pi}\, \sin(\theta)$ $\phi = 2\pi\, u$ $\theta = \arccos(1 - v)$		
Uniform sampling of a hemisphere		
$p(\theta, \phi) = \frac{1}{\pi}\, \cos(\theta)\, \sin(\theta)$ $\phi = 2\pi\, u$ $\theta = \arcsin(\sqrt{v})$		
Uniform sampling of a cosine-weighted hemisphere		

[5] M. Pharr, G. Humphreys: Physically Based Rendering (2010), ch. 7.4.2, pp. 361ff

Table 2: Summary of mappings (continued)

PDF / Mapping	Top view	Front and back view
$p(\theta, \phi) = \frac{n+1}{2\pi}\ \cos^n(\theta)\ \sin(\theta)$ $\phi = 2\pi\, u$ $\theta = \arccos\left((1-v)^{\frac{1}{n+1}}\right)$		
Phong (or Blinn-Phong) specular lobe with exponent $n = 40$		
$p(\theta, \phi) = D(\theta)\cos(\theta)\sin(\theta)$ $\phi = 2\pi\, u$ $\theta = \arccos\left(\sqrt{\frac{1-v}{v(\alpha^2-1)+1}}\right)$		
Microfacet GGX specular lobe with roughness $\alpha = r^2$ and $r = 0.5$		
$p(\theta, \phi) = D(\theta)\cos(\theta)\sin(\theta)$ $\phi = 2\pi\, u$ $\theta = \arccos\left(\sqrt{\frac{1-v}{v(\alpha^2-1)+1}}\right)$		
Microfacet GGX specular lobe with roughness $\alpha = r^2$ and $r = 0.25$		