$$\mathbb{I} \text{ komm unit ring}$$

$$\mathbb{I} \supset \underset{\text{mult}}{\mathbb{U}} \supset \mathbb{U} \times \mathbb{U}$$

$$\mathbb{I} \supset \mathbb{U} \ni e \text{ unit}$$

$$R \text{ relation on } \mathbb{I} \times \mathbb{U}$$

$$m{:}p \sim n{:}q \Leftrightarrow \bigvee_{b \in \mathbb{U}} \underbrace{m \times q} \times b = \underbrace{n \times p} \times b \text{ equ rel}$$

$$\text{refl } m \times p = m \times p \Leftrightarrow m{:}p \sim m{:}p$$

$$\text{symm } m{:}p \sim n{:}q \Leftrightarrow m \times q \times b = n \times p \times b \Leftrightarrow n \times p \times b = m \times q \times b \Leftrightarrow n{:}q \sim m{:}p$$

$$\text{trans } m{:}p \sim n{:}q \sim r{:}s \Leftrightarrow \begin{cases} \bigvee_{b \in \mathbb{U}} & m \times q \times b = n \times p \times b \\ \bigvee_{d \in \mathbb{U}} & n \times s \times d = r \times q \times d \end{cases}$$

$$\Leftrightarrow \underbrace{m \times s} \times \underbrace{q \times b \times d}_{\in \mathbb{U} \text{ mult}} = \underbrace{m \times q \times b} \times \underbrace{s \times d} \underset{\text{Vor}}{=} \underbrace{n \times p \times b} \times \underbrace{s \times d}$$

$$= \underbrace{n \times s \times d} \times \underbrace{p \times b} \underset{\text{Vor}}{=} \underbrace{r \times q \times d} \times \underbrace{p \times b} = \underbrace{r \times p} \times \underbrace{q \times b \times d} \Leftrightarrow m{:}p \sim r{:}s$$

$$\text{quotient class } m \oslash p = \frac{m}{p} = \frac{n{:}q}{m{:}p \sim n{:}q}$$

$$m{:}p \sim n{:}q \Leftrightarrow m \oslash p \sim n \oslash q \Leftrightarrow \frac{m}{p} = \frac{n}{q}$$

$$\mathbb{I} \oslash \mathbb{U} = \underbrace{\mathbb{I} \times \mathbb{U}} \ulcorner R = \begin{cases} m \oslash p = \dfrac{m}{p} \\ m \in \mathbb{I}: \quad p \in \mathbb{U} \end{cases}$$

$$m \oslash p \in \mathbb{I} \oslash \mathbb{U} \xleftarrow[\text{surj}]{\oslash} \mathbb{I} \times \mathbb{U} \ni m{:}p$$

$$\text{add ratios } \underbrace{\mathbb{I} \oslash \mathbb{Y}} \times \underbrace{\mathbb{I} \oslash \mathbb{Y}} \xrightarrow{+} \mathbb{I} \oslash \mathbb{Y}$$

$$\underbrace{m \oslash p} + \underbrace{n \oslash q} = \underbrace{m \times q + n \times p} \oslash \underbrace{p \times q}$$

$$\frac{m}{p} + \frac{n}{q} = \frac{m \times q + n \times p}{p \times q} \text{ well-def}$$

$$\begin{cases} m \oslash p = \acute{m} \oslash \acute{p} & \Leftrightarrow \bigvee_{b \in \mathbb{Y}} \underbrace{m \times \acute{p}} \times b = \underbrace{\acute{m} \times p} \times b \\ n \oslash q = \acute{n} \oslash \acute{q} & \Leftrightarrow \bigvee_{d \in \mathbb{Y}} \underbrace{n \times \acute{q}} \times d = \underbrace{\acute{n} \times q} \times d \end{cases}$$

$$\Leftrightarrow \overbrace{\underbrace{m \times q + n \times p} \times \underbrace{\acute{p} \times \acute{q}}} \times \underbrace{b \times d}_{\in \mathbb{Y} \text{ mult}} = \underbrace{m \times q} \times \underbrace{\acute{p} \times \acute{q}} \times \underbrace{b \times d} + \underbrace{n \times p} \times \underbrace{\acute{p} \times \acute{q}} \times \underbrace{b \times d}$$

$$= \underbrace{m \times \acute{p} \times b} \times \underbrace{q \times \acute{q} \times d} + \underbrace{n \times \acute{q} \times d} \times \underbrace{p \times \acute{p} \times b} \underset{\text{Vor}}{=} = \underbrace{\acute{m} \times p \times b} \times \underbrace{q \times \acute{q} \times d} + \underbrace{\acute{n} \times q \times d} \times \underbrace{p \times \acute{p} \times b}$$

$$= \underbrace{\acute{m} \times \acute{q}} \times \underbrace{p \times q} \times \underbrace{b \times d} + \underbrace{\acute{n} \times \acute{p}} \times \underbrace{p \times q} \times \underbrace{b \times d} = \overbrace{\underbrace{\acute{m} \times \acute{q} + \acute{n} \times \acute{p}} \times \underbrace{p \times q}} \times \underbrace{b \times d}$$

$$\Leftrightarrow \underbrace{m \times q + n \times p} : \underbrace{p \times q} \sim \underbrace{\acute{m} \times \acute{q} + \acute{n} \times \acute{p}} : \underbrace{\acute{p} \times \acute{q}} \Leftrightarrow \underbrace{m \times q + n \times p} \oslash \underbrace{p \times q} = \underbrace{\acute{m} \times \acute{q} + \acute{n} \times \acute{p}} \oslash \underbrace{\acute{p} \times \acute{q}}$$

$$\text{multiply ratios } \underbrace{\mathbb{I} \oslash \mathbb{Y}} \times \underbrace{\mathbb{I} \oslash \mathbb{Y}} \xrightarrow{\times} \mathbb{I} \oslash \mathbb{Y}$$

$$\underbrace{m \oslash p} \times \underbrace{n \oslash q} = \underbrace{m \times n} \oslash \underbrace{p \times q}$$

$$\frac{m}{p} \times \frac{n}{q} = \frac{m \times n}{p \times q} \text{ well-def}$$

$$\begin{cases} m \oslash p = \acute{m} \oslash \acute{p} & \Leftrightarrow m \times \acute{p} \times b = \acute{m} \times p \times b \\ n \oslash q = \acute{n} \oslash \acute{q} & \Leftrightarrow n \times \acute{q} \times d = \acute{n} \times q \times d \end{cases} \Leftrightarrow$$

$$\overbrace{\underbrace{m \times n} \times \underbrace{\acute{p} \times \acute{q}}} \times \underbrace{b \times d}_{\in \mathbb{Y} \text{ mult}} = \underbrace{m \times \acute{p} \times b} \times \underbrace{n \times \acute{q} \times d} \underset{\text{Vor}}{=} \underbrace{\acute{m} \times p \times b} \times \underbrace{\acute{n} \times q \times d} = \overbrace{\underbrace{\acute{m} \times \acute{n}} \times \underbrace{p \times q}} \times \underbrace{b \times d}$$

$$\Leftrightarrow \underbrace{m \times n} : \underbrace{p \times q} \sim \underbrace{\acute{m} \times \acute{n}} : \underbrace{\acute{p} \times \acute{q}} \Leftrightarrow \underbrace{m \times n} \oslash \underbrace{p \times q} = \underbrace{\acute{m} \times \acute{n}} \oslash \underbrace{\acute{p} \times \acute{q}}$$

$$\text{triv ratio} \quad \begin{tikzcd} m \oslash e \in \mathbb{I} \oslash \mathbb{Y} & \mathbb{I} \times \mathbb{Y} \arrow[l, "\oslash"'] & \mathbb{I} \ni m \arrow[l, ":e"'] \arrow[ll, bend left, "\oslash e"] \end{tikzcd}$$

$$\mathbb{Y} \text{ kuerzbar} \Leftrightarrow \text{inj}$$

$$/ \qquad \underbrace{m \oslash e} + \underbrace{n \oslash e} = \underbrace{m + n} \oslash e$$

$$\underbrace{m \oslash e} + \underbrace{n \oslash e} = \underbrace{m \times e + n \times e} \oslash \underbrace{e \times e} = \underbrace{m + n} \oslash e$$

$$/ \qquad \underbrace{m \oslash e} \times \underbrace{n \oslash e} = \underbrace{m \times n} \oslash e$$

$$\underbrace{m \oslash e} \times \underbrace{n \oslash e} = \underbrace{m \times n} \oslash \underbrace{e \times e} = \underbrace{m \times n} \oslash e$$