

$$\text{order } \mathfrak{h}:P \in \mathbb{N}$$

$$\ell \begin{array}{c} \mathfrak{h} \\ \mathbb{N} \end{array} = \frac{\ell \xrightarrow{\mathfrak{h}} \mathfrak{h} \text{ strong isoton}}{0 \mathfrak{h} < 1 \mathfrak{h} < \dots < \ell^{-1} \mathfrak{h}} = \underbrace{\ell \begin{array}{c} \mathfrak{h} \\ \mathbb{N} \end{array}}_{\text{isoton}} \cap \underbrace{\ell \begin{array}{c} \mathfrak{h} \\ \mathbb{N} \end{array}}_{\text{inj}}$$

$$1 \leq \ell \Rightarrow \overbrace{x}^{\ell} \overbrace{P-I}^y = \#^{\ell-1} \begin{array}{c} \mathfrak{h} \\ \mathbb{N} \end{array} x|y$$

$$\overbrace{x}^{\ell} \overbrace{P-I}^y = \overbrace{x}^{\ell} P^y - \overbrace{x}^{\ell} I^y = \begin{cases} 0 & x=y \\ 1 & x < y \end{cases} = \#^0 \begin{array}{c} \mathfrak{h} \\ \mathbb{N} \end{array} x|y$$

$$\overbrace{x}^{\ell} \overbrace{P-I}^y \ast \overbrace{x}^{\ell} \overbrace{P-I}^y = \sum_{x \leq z \leq y} \overbrace{x}^{\ell} \overbrace{P-I}^z \overbrace{z}^{\ell} \overbrace{P-I}^y = \sum_{x < z < y} 1 = \#^1 \begin{array}{c} \mathfrak{h} \\ \mathbb{N} \end{array} x|y$$

$$\overbrace{x}^{\ell} \overbrace{P-I}^y = \sum_{x = z_0 < z_1 < \dots < z_{\ell-1} < z_{\ell} = y} 1 = \#^{\ell-1} \begin{array}{c} \mathfrak{h} \\ \mathbb{N} \end{array} x|y$$

$$\overbrace{x}^{\ell} \overbrace{1+tI-tP}^y = \sum_{1 \leq \ell} t^{\ell} \#^{\ell-1} \begin{array}{c} \mathfrak{h} \\ \mathbb{N} \end{array} x|y$$

$$\overbrace{1+tI-tP}^y = \overbrace{I-tP-I}^y = \sum_{1 \leq \ell} \overbrace{tP-I}^y = \sum_{1 \leq \ell} t^{\ell} \overbrace{P-I}^y \Rightarrow \text{LHS} = \sum_{1 \leq \ell} t^{\ell} \overbrace{x}^{\ell} \overbrace{P-I}^y = \text{RHS}$$

$$\text{Moebius } \overbrace{x}^{\ell} \overbrace{P-I}^y = \sum_{1 \leq \ell} (-1)^{\ell} \#^{\ell-1} \begin{array}{c} \mathfrak{h} \\ \mathbb{N} \end{array} x|y$$

$$t = -1$$