

$$\begin{aligned} \mathbb{H} \leqslant \mathbb{H}_{\Delta \mathbb{K}} &= \begin{cases} \mathbb{H} \times \mathbb{H} \xrightarrow{\sim} \mathbb{K} \\ \text{Trg } \mathcal{N} \subset P \\ {}_x \mathcal{V}^y \neq 0 \curvearrowright x:y \in P \end{cases} \\ \mathcal{N} * \mathfrak{A} \in \mathbb{H} \leqslant \mathbb{H}_{\Delta \mathbb{K}} &\leftarrow \underset{\text{bilin}}{*} \quad \mathbb{H} \leqslant \mathbb{H}_{\Delta \mathbb{K}} \times \mathbb{H} \leqslant \mathbb{H}_{\Delta \mathbb{K}} \ni \mathcal{N} : \mathfrak{A} \end{aligned}$$

$${}_x \underbrace{\mathcal{N} * \mathfrak{A}}^z = \sum_y^{x|z} {}_x \mathcal{V}^y {}_y \mathfrak{A}^z$$

$$x:z \notin P \Rightarrow x|z = \emptyset \underset{\text{void sum}}{\Rightarrow} {}_x \underbrace{\mathcal{N} * \mathfrak{A}}^z = 0$$

$$\underbrace{\mathcal{N} * \mathfrak{A} * \mathfrak{B}}_{\text{assoc}} = \underbrace{\mathcal{N} * \mathfrak{B} * \mathfrak{A}}$$

$$\begin{aligned} {}_x \overbrace{\mathcal{N} * \mathfrak{A} * \mathfrak{B}}^w &= \sum_z^{x|w} {}_x \overbrace{\mathcal{N} * \mathfrak{A}}^z {}_z \mathfrak{B}^w = \sum_{x \leqslant y \leqslant z \leqslant w} {}_x \mathcal{V}^y {}_y \mathfrak{A}^z {}_z \mathfrak{B}^w = \sum_z^{x|w} \sum_y^{x|z} \underbrace{{}_x \mathcal{V}^y {}_y \mathfrak{A}^z}_{} {}_z \mathfrak{B}^w \\ &= \sum_y^{x|w} \sum_z^{y|w} {}_x \mathcal{V}^y \underbrace{{}_y \mathfrak{A}^z {}_z \mathfrak{B}^w}_{} = \sum_y^{x|w} {}_x \mathcal{V}^y \overbrace{{}_y \mathfrak{A} * \mathfrak{B}}^w = {}_x \overbrace{\mathcal{N} * \mathfrak{B} * \mathfrak{A}}^w \end{aligned}$$

$$\mathbb{H} \leqslant \mathbb{H}_{\Delta \mathbb{K}} \ni \mathcal{N} \text{ inv} \Leftrightarrow \begin{cases} \bigvee \mathcal{N}^{-1} \in \mathbb{H} \leqslant \mathbb{H}_{\Delta \mathbb{K}} \\ \mathcal{N} * \mathcal{N}^{-1} = I = \mathcal{N}^{-1} * \mathcal{N} \end{cases}$$

$$\text{inv } \mathfrak{N} \in \mathbb{H} \leqslant \mathbb{H} \Delta \mathbb{K} \Leftrightarrow \bigwedge_x {}_x \mathfrak{N}^x \neq 0$$

$$x < y \Rightarrow \begin{cases} {}_x \mathfrak{C}^x {}_x \bar{\mathfrak{C}}^y = - \sum_{x < z \leqslant y} {}_x \mathfrak{C}^z {}_z \bar{\mathfrak{C}}^y \\ \sum_{x \leqslant z \leqslant y} {}_x \mathfrak{C}^z {}_z \bar{\mathfrak{C}}^y = 0 \end{cases}$$

$$\Rightarrow : 1 = {}_x I^x = \overbrace{{}_x \mathfrak{C} * {}_x \bar{\mathfrak{C}}}^x = \sum_y {}_x \mathfrak{C}^y {}_y \bar{\mathfrak{C}}^x = {}_x \mathfrak{C}^x {}_x \bar{\mathfrak{C}}^x \Rightarrow {}_x \mathfrak{C}^x \neq 0$$

$$\Leftarrow : {}_x \bar{\mathfrak{C}}^y \text{ ind def } n = \#x|y \geqslant 1$$

$$\#x|y = 1 \Rightarrow x = y \Rightarrow {}_x \bar{\mathfrak{C}}^x = \frac{1}{{}_x \mathfrak{C}}$$

$$1 < \#x|y \Rightarrow \bigwedge_{x < z \leqslant y} \#z|y < \#x|y \underset{\text{ind}}{\Rightarrow} {}_z \bar{\mathfrak{C}}^y \text{ well-def}$$

$${}_x \bar{\mathfrak{C}}^y \underset{*}{=} -\frac{1}{{}_x \mathfrak{C}} \sum_{x < z \leqslant y} {}_x \mathfrak{C}^z {}_z \bar{\mathfrak{C}}^y$$

$$\Rightarrow \overbrace{{}_x \mathfrak{C} * {}_x \bar{\mathfrak{C}}}^y = \sum_{x \leqslant z \leqslant y} {}_x \mathfrak{C}^z {}_z \bar{\mathfrak{C}}^y = {}_x \mathfrak{C}^x {}_x \bar{\mathfrak{C}}^y + \sum_{x < z \leqslant y} {}_x \mathfrak{C}^z {}_z \bar{\mathfrak{C}}^y \underset{*}{=} - \sum_{x < z \leqslant y} {}_x \mathfrak{C}^z {}_z \bar{\mathfrak{C}}^y + \sum_{x < z \leqslant y} {}_x \mathfrak{C}^z {}_z \bar{\mathfrak{C}}^y = 0 \underset{x < y}{=} {}_x I^y$$