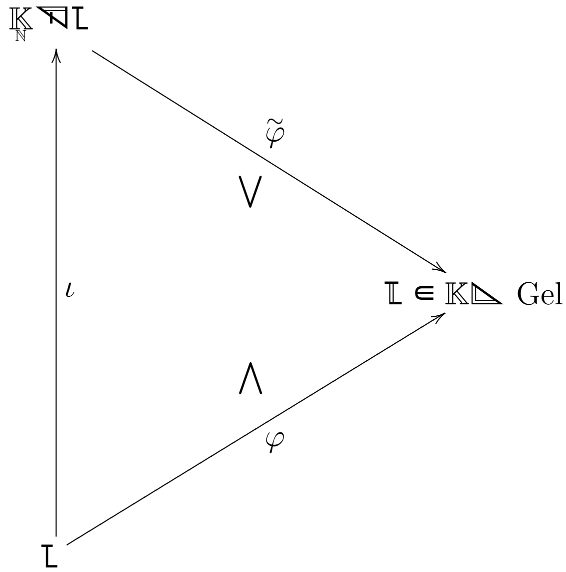
$$\mathbb{K}_m \triangledown \rceil \rtimes C|m \longrightarrow \mathbb{K}_m \triangledown \rceil$$

$$\overbrace{{}_1\rceil \boxtimes \cdots \boxtimes {}_m\rceil} \rtimes \sigma = {}_{\sigma_1}\rceil \boxtimes \cdots \boxtimes {}_{\sigma_m}\rceil$$

$$\mathbb{K}_{\mathbb{N}} \triangledown \rceil = \sum_m \mathbb{K}_m \triangledown \rceil \in \mathbb{K}\triangle \text{ abel}$$



$$\lfloor \varphi \ast \lfloor' \varphi = 0$$

$$\lfloor \iota = \lfloor + \mathcal{J}$$

$$\lfloor \iota \ast \lfloor' \iota = 0$$

$$\left({}_1\lfloor \boxtimes \cdots \boxtimes {}_m\lfloor + \mathcal{J} \right) \tilde{\varphi} := {}_1\lfloor \varphi \times \cdots \times {}_m\lfloor \varphi$$

$$\left({}_1\lfloor \boxtimes \cdots \boxtimes {}_m\lfloor + \mathcal{J} \right) \mathbb{K}_{\mathbb{N}} \triangledown \rceil := {}_1\lfloor \rceil \boxtimes \cdots \boxtimes {}_m\lfloor \rceil + \mathcal{J}$$

$$\mathbb{K}_{\mathbb{N}} \triangledown \rceil := \mathbb{K}_{\mathbb{N}} \triangledown \rceil \ulcorner \mathbb{K}_{\mathbb{N}} \triangledown \rceil \left(\lfloor \boxtimes \lfloor' - \lfloor' \boxtimes \lfloor \right) \mathbb{K}_{\mathbb{N}} \triangledown \rceil$$

$$\mathbb{K}_m \triangledown \rceil := \mathbb{K}_m \triangledown \rceil + \mathbb{K}\ {}_{\pi_1}\lfloor \boxtimes \cdots \boxtimes {}_{\pi_1}\lfloor - \overset{\pi}{-}1\ {}_1\lfloor \boxtimes \cdots \boxtimes {}_m\lfloor$$

$$_I\lfloor = \mathop{\boxtimes}_{i \in I} {}_i\lfloor = \begin{bmatrix} {}_{i_1}\lfloor \\ + \\ {}_{i_m}\lfloor \end{bmatrix}$$

$$I = \{ i_1 \leqslant \cdots \leqslant i_m \} \text{ geordnet}$$

$$\mathbb{K}_{\mathbb{N}} \triangledown \rceil \in \mathbb{K}\triangle_0^{1:1}$$

$$\llcorner_I \ast \llcorner'_J = \bigotimes_{i \in I} \llcorner_i \ast \bigotimes_{j \in J} \llcorner'_j = \operatorname{per} \llcorner_i \ast \llcorner'_j = \operatorname{per} \llcorner_i \eta \left(\overset{*}{\llcorner'_j} \right)$$

$$\mathbb{K}_m \lrcorner \to \mathbb{K} \lrcorner \, \overset{m}{\mathbb{K}}$$

$$\bigotimes_{i \in I} \llcorner_i \,|\, \bigotimes_{j \in J} \urcorner^j = \operatorname{per} \llcorner_i \urcorner^j$$

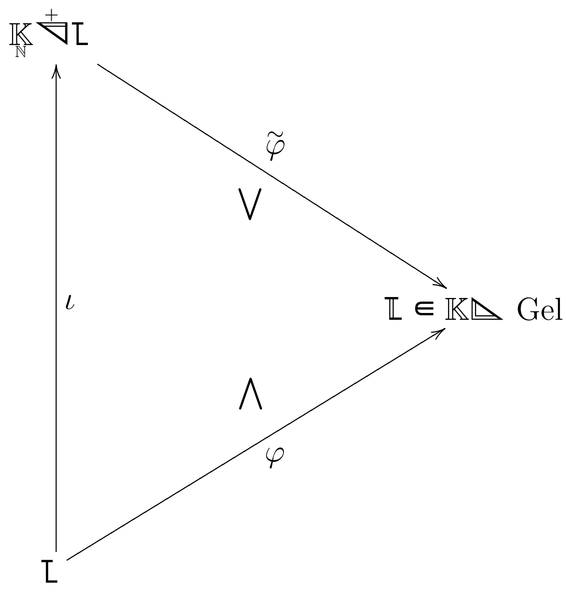
$$\text{LHS} = m! \bigotimes_{i \in I} \llcorner_i \,|\, p_- \bigotimes_{j \in J} \urcorner^j = \sum_{I \xrightarrow{\asymp} \pi J} \overset{\pi}{-1} \bigotimes_{i \in I} \llcorner_i \bigotimes_{i \in I} \urcorner^{\pi_i} = \sum_{\pi} \prod_{i \in I} \llcorner_i \urcorner^{\pi_i} = \text{RHS}$$

$$\mathbb{I}_{\mathbb{N}} \lrcorner = \mathbb{I} \otimes \mathbb{K}_{\mathbb{N}} \lrcorner$$

$$_N\llcorner \in \mathbb{K}_n \lrcorner$$

$$_N\llcorner \; \lrcorner = \; _N\llcorner \; \operatorname{perm} \; \lrcorner$$

$$\mathbb{K}_{\mathbb{N}}^{+} \lrcorner = \sum_m \mathbb{K}_m^{+} \lrcorner \in \mathbb{K} \; \text{abel}$$



$$\llcorner \varphi \times \llcorner' \varphi = 0$$

$$\llcorner \iota = \llcorner + \mathcal{J}$$

$$\llcorner \iota \times \llcorner' \iota = 0$$

$$\left({}_1\llcorner \otimes \cdots \otimes {}_m\llcorner + \mathcal{J} \right) \widetilde{\varphi} := {}_1\llcorner \, \varphi \times \cdots \times {}_m\llcorner \, \varphi$$

$$\left({}_1v \otimes \cdots \otimes {}_m v + \mathcal{J} \right) \underset{\mathbb{N}}{\mathbb{K}} \Lambda V := {}_1v \; V \otimes \cdots \otimes {}_m v \; V + \mathcal{J}$$

$$\underset{\mathbb{N}}{\mathbb{K}} \overset{+}{\Lambda} V := \underset{\mathbb{N}}{\mathbb{K}} \Lambda V + \underset{\mathbb{N}}{\mathbb{K}} \Lambda V \left\{ v \otimes v' - v' \otimes v \right\} \underset{\mathbb{N}}{\mathbb{K}} \Lambda V$$

$$\underset{m}{\mathbb{K}} \overset{+}{\Lambda} V = \underset{m}{\mathbb{K}} \Lambda V + \mathbb{K} \overbrace{{}_{\pi_1} v \otimes \cdots \otimes {}_{\pi_1} v - {}_1 v \otimes \cdots \otimes {}_m v}$$

$${}_I v = \bigotimes_{i \in I} {}_i v = \begin{bmatrix} {}_{i_1} v \\ \vdots \\ {}_{i_m} v \end{bmatrix}$$

$$I = \left\{ i_1 \leqslant \cdots \leqslant i_m \right\} \text{ geordnet}$$

$$\underset{\mathbb{N}}{\mathbb{K}} \overset{+}{\Lambda} V \in \mathbb{K} \overset{1:1}{\underset{0}{\Delta}}$$

$${}_I v \wedge {}_J v' = \bigotimes_{i \in I} {}_i v \wedge \bigotimes_{j \in J} {}_j v' = \text{ Per } {}_i v \wedge {}_j v' = \text{ Per } {}_i v \, \eta \left(\overset{*}{{}_j v'} \right)$$

$$\underset{m}{\mathbb{K}} \overset{+}{\Lambda} V \to \mathbb{K} \Lambda V \underset{\Delta}{\overset{+ \; m}{}} \mathbb{K}$$

$$\underset{m}{\mathbb{K}} \overset{+}{\Lambda} V_{\mathbb{K} \Delta} \leftarrow V \underset{\Delta}{\overset{+ \; m}{}} \mathbb{K}$$

$$\bigotimes_{i \in I} {}_i v \, | \bigotimes_{j \in J} e^j = \text{ Per } {}_i v \, e^j$$

$$\text{LHS } = m! \bigotimes_{i \in I} {}_i v \, | p_+ \bigotimes_{j \in J} e^j = \sum_{I \overset{\asymp}{\longrightarrow} \pi J} \bigotimes_{i \in I} {}_i v \bigotimes_{i \in I} e^{\pi_i} = \sum_{\pi} \prod_{i \in I} {}_i v \, e^{\pi_i} = \text{ RHS}$$

$$\underset{\mathbb{N}}{T} \overset{+}{\Lambda} V = T \otimes \underset{\mathbb{N}}{\mathbb{K}} \overset{+}{\Lambda} V$$

$$V \underset{\Delta}{\overset{+ \; \mathbb{N}}{}} \mathbb{K} = \sum_m V \underset{\Delta}{\overset{+ \; m}{}} \mathbb{K} \in \overset{\mathbb{N}}{\Lambda} \mathbb{K} \text{ Gel}$$

$$V \underset{\Delta}{\overset{+ \; m}{}} \mathbb{K} = \frac{V \otimes \cdots \otimes V \xrightarrow{\text{m-lin}} T\mathbb{K}}{\pi \ltimes T = T} = \underset{m}{\mathbb{K}} \overset{+}{\Lambda} V_{\Delta} \mathbb{K} \text{ symm}$$

$$V \underset{\Delta}{\overset{p+q}{}} \mathbb{K} \overset{\otimes}{\longleftarrow} V \underset{\Delta}{\overset{p}{}} \mathbb{K} \otimes V \underset{\Delta}{\overset{q}{}} \mathbb{K}$$

$${}_M v \left(\overset{p}{T} \otimes T' \right) = \sum_{P \subset M}^{\sharp P = p} \left({}_P v \, T \right) \left({}_{M \llcorner P} v \, T' \right)$$

$$\begin{bmatrix} {}_1 \\ + \\ {}_m \end{bmatrix} \otimes ' = \sum_{1 \leqslant \nu_1 < \cdots < \nu_p \leqslant m} \begin{bmatrix} \nu_1 \\ + \\ \nu_p \end{bmatrix} \begin{bmatrix} \nu_{p+1} \\ + \\ \nu_m \end{bmatrix} '$$

$$\mathbb{K}_m^{+} \; \mathbb{K} \xleftarrow{?} {}^{+\,m}\mathbb{K}$$

$$\otimes_{i \in I} {}_i \,|\, \otimes_{j \in J} {}^j = \text{Per} \; {}_i \, {}^j$$

$$\text{LHS} = m! \otimes_{i \in I} {}_i \,|\, p_+ \otimes_{j \in J} {}^j = \sum_{I \xrightarrow{\asymp} \pi J} \otimes_{i \in I} {}_i \otimes_{i \in I} {}^{\pi_i} = \sum_{\pi} \prod_{i \in I} {}_i \, {}^{\pi_i} = \text{RHS}$$

$$ {}^{+\,\mathbb{N}}\mathbb{1} = \sum_m {}^{+\,m}\mathbb{1}$$

$$ {}^{+\,m}\mathbb{1} = \left\{ \otimes \cdots \otimes \xrightarrow{\text{m-lin}} \mathbb{1} \;\; \middle| \;\; \pi \ltimes = \right\} = \mathbb{K}_m^{+} \mathbb{1} = \underbrace{{}^{+\,m}\mathbb{K}} \otimes \mathbb{1}$$

$$ {}''^{\,p+q}\mathbb{1} \xleftarrow{\otimes} {}^{p}\mathbb{1} \otimes {}'^{\,q}\mathbb{1}$$

$$ {}^{+\,\mathbb{N}}\mathbb{1} = \sum_m {}^{+\,m}\mathbb{1} \in {}^{\mathbb{N}}\mathbb{K} \text{ Gel}$$

$$ {}^{+\,\mathbb{N}}\mathbb{K} \underset{p_+}{\overset{\sqsubset}{\rightleftarrows}} {}^{\mathbb{N}}\mathbb{K}$$

$$p_+ = \frac{1}{m!} \sum_{\pi \in \mathsf{C}|m} \pi \ltimes$$

$$p_+ (\tau \ltimes) = p_+$$

$$p_+ \left(p_+ \right) = p_+$$

$$ {}^{+\,\mathbb{N}}\mathbb{1} \underset{p_+}{\overset{\sqsubset}{\rightleftarrows}} {}^{\mathbb{N}}\mathbb{1}$$

$$\left(p_{+}\mathbb{1}\right)\boxtimes\left(p_{+}\mathbb{1}'\right)=\begin{bmatrix}m\\pq\end{bmatrix}p_{+}\left(\mathbb{1}\boxtimes\mathbb{1}'\right)$$

$$m!\,{}_{M}\llcorner\, p_{+}\left(\mathbb{1}\boxtimes\mathbb{1}'\right)=\sum_{\pi\in\mathsf{C}|M}{}_{\pi_M}\llcorner\left(\mathbb{1}\boxtimes\mathbb{1}'\right)=\sum_{\pi\in\mathsf{C}|M}\left({}_{\pi_{M_<}}\llcorner\mathbb{1}\right)\times\left({}_{\pi_{M_>}}\llcorner\mathbb{1}'\right)$$

$$=\sum_{|P|=p}\sum_{M_<\xrightarrow{\asymp}\sigma P}\left({}_{\sigma}\llcorner\mathbb{1}\right)\times\sum_{M_>\xrightarrow{\asymp}\tau M\llcorner P}\left({}_{\tau}\llcorner\mathbb{1}'\right)=\sum_{|P|=p}p!\left({}_{P}\llcorner p_{+}\mathbb{1}\right)\times q!\left({}_{M\llcorner P}\llcorner p_{+}\mathbb{1}'\right)$$

$$p_{+}\left(\left(p_{+}\mathbb{1}\right)\boxtimes\mathbb{1}'\right)=p_{+}\left(\mathbb{1}\boxtimes\mathbb{1}'\right)=p_{+}\mathbb{1}\boxtimes\left(p_{+}\mathbb{1}'\right)$$

$$\begin{bmatrix}m\\pq\end{bmatrix}p_{+}\left(\left(p_{+}\mathbb{1}\right)\boxtimes\mathbb{1}'\right)=p_{+}\left(\left(p_{+}\mathbb{1}\right)\boxtimes\left(p_{+}\mathbb{1}'\right)\right)=\left(p_{+}\mathbb{1}\right)\boxtimes\left(p_{+}\mathbb{1}'\right)=\begin{bmatrix}m\\pq\end{bmatrix}p_{+}\mathbb{1}\boxtimes\left(p_{+}\mathbb{1}'\right)$$

$$\llcorner^{+}_{\triangle}\mathbb{K}^{\mathbb{N}}\ \text{assoc comm}$$

$$\left(\mathbb{1}\boxtimes\mathbb{1}'\right)\boxtimes\mathbb{1}''=\begin{bmatrix}m\\p+q{:}r\end{bmatrix}p_{+}\left(\left(\mathbb{1}\boxtimes\mathbb{1}'\right)\boxtimes\mathbb{1}''\right)=\begin{bmatrix}m\\p{:}q{:}r\end{bmatrix}p_{+}p_{+}\left(\mathbb{1}\boxtimes\mathbb{1}'\right)\boxtimes\mathbb{1}''=\begin{bmatrix}m\\p{:}q{:}r\end{bmatrix}p_{+}\left(\mathbb{1}\boxtimes\mathbb{1}'\boxtimes\mathbb{1}''\right)$$

$$\mathbb{1}\boxtimes\mathbb{1}'=\begin{bmatrix}m\\pq\end{bmatrix}p_{+}\left(\mathbb{1}\boxtimes\mathbb{1}'\right)=\begin{bmatrix}m\\pq\end{bmatrix}p_{+}\tau\ltimes\left(\mathbb{1}'\boxtimes\mathbb{1}\right)=\left(\mathbb{1}'\boxtimes\mathbb{1}\right)$$

$$\tau\begin{bmatrix}M_<\\M_>\end{bmatrix}=\begin{bmatrix}M_>\\M_<\end{bmatrix}$$

$$\mathbb{1}^{0}\boxtimes\cdots\boxtimes\mathbb{1}^{k}=\begin{bmatrix}m\\p_0\cdots p_k\end{bmatrix}p_{+}\mathbb{1}^{0}\boxtimes\cdots\boxtimes\mathbb{1}^{k}$$

$$\text{LHS}=\mathbb{1}^{0}\boxtimes\left(\mathbb{1}^{1}\boxtimes\cdots\boxtimes\mathbb{1}^{k}\right)=\begin{bmatrix}m\\p_0{:}p_1\cdots p_k\end{bmatrix}p_{+}\mathbb{1}^{0}\boxtimes\left(\mathbb{1}^{1}\boxtimes\cdots\boxtimes\mathbb{1}^{k}\right)=\begin{bmatrix}m\\p_0{:}p_1\cdots p_k\end{bmatrix}\begin{bmatrix}p_1+\cdots+p_k\\p_1\cdots p_k\end{bmatrix}p_{+}\mathbb{1}^{0}\boxtimes\left(p_{+}\mathbb{1}^{1}\boxtimes\cdots\boxtimes\mathbb{1}^{k}\right)=$$

$$\llcorner^{+}_{\triangle}\mathbb{K}^{m}=\llcorner_{\triangle}\mathbb{K}\,{}^{+}_{\triangle}\mathbb{K}^{m}\ni\mathbb{1}^{J}=\boxtimes_{j\in J}\mathbb{1}^{j}$$

$$(\boxtimes\mathbb{L})\boxtimes\left(\boxtimes\mathbb{L}'\right):=\mathbb{L}\boxtimes\mathbb{L}'$$

$$(\bowtie \mathbb{L}) \bowtie \mathbb{T} = \mathbb{L}\mathbb{T}$$

$$\bowtie_I \lfloor = \bowtie_{i \in I} (\bowtie_i \lfloor)$$

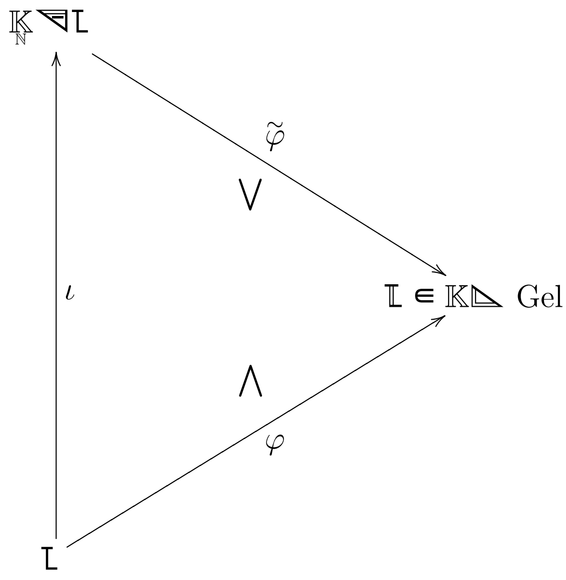
$$(\bowtie_I \lfloor) \bowtie \left(\bowtie_J \lfloor'\right) = {}_I\lfloor \bowtie {}_J\lfloor'$$

$$\rceil^I \bowtie \rceil'^J = \left(\bowtie_{i \in I} \rceil^i\right) \bowtie \left(\bowtie_{j \in J} \rceil'^j\right) = \text{Per}\left(\rceil^i \bowtie \rceil'^j\right)$$

$$\left(\bowtie_{i \in I} {}_i\lfloor\right) \bowtie \left(\bowtie_{j \in J} {}_j\lfloor'\right) = \text{Per}\left({}_i\lfloor \bowtie {}_j\lfloor'\right) = \text{Per}\left({}_i\lfloor \left(\bowtie {}_j\lfloor'\right)\right) = \left(\bowtie_{i \in I} {}_i\lfloor\right)\left(\bowtie_{j \in J} \left(\bowtie {}_j\lfloor'\right)\right)$$

$$(\bowtie(\bowtie {}_i\lfloor)) \bowtie \left(\bowtie \rceil^j\right) = (\bowtie(\bowtie {}_i\lfloor)) \bowtie \left(\bowtie \rceil^j\right) = (\bowtie {}_i\lfloor)\left(\bowtie \rceil^j\right) = \text{Per}\left({}_i\lfloor \rceil^j\right) = \text{Per}\left((\bowtie {}_i\lfloor) \bowtie \rceil^j\right)$$

$$\mathbb{K}_{\mathbb{N}} \triangledown \mathbb{L} = \sum_m \mathbb{K}_m \triangledown \mathbb{L} \in \mathbb{K}\triangle\text{-abel}$$



$$\lfloor\varphi \ast \lfloor'\varphi = 0$$

$$\lfloor\iota = \lfloor + \mathcal{J}$$

$$\lfloor\iota \ast \lfloor'\iota = 0$$

$$({}_1\lfloor \bowtie \cdots \bowtie {}_m\lfloor + \mathcal{J})\,\widetilde{\varphi} := {}_1\lfloor\varphi \times \cdots \times {}_m\lfloor\varphi$$

$$({}_1\lfloor \bowtie \cdots \bowtie {}_m\lfloor + \mathcal{J})\,\mathbb{K}_{\mathbb{N}} \triangledown \mathbb{L} := {}_1\lfloor\ \mathbb{L} \bowtie \cdots \bowtie {}_m\lfloor\ \mathbb{L} + \mathcal{J}$$

$$\mathbb{K}_{\mathbb{N}} \triangledown \mathbb{L} := \mathbb{K}_{\mathbb{N}} \triangledown \mathbb{L} + \mathbb{K}_{\mathbb{N}} \triangledown \mathbb{L}\,\lfloor \bowtie \lfloor' + \lfloor' \bowtie \lfloor\,\mathbb{K}_{\mathbb{N}} \triangledown \mathbb{L}$$

$$\mathbb{K}_m \triangledown \mathbb{L} := \mathbb{K}_m \triangledown \mathbb{L} + \mathbb{K}\ {}_{\pi_1}\lfloor \bowtie \cdots \bowtie {}_{\pi_1}\lfloor - \overset{\pi}{-}1\ {}_1\lfloor \bowtie \cdots \bowtie {}_m\lfloor$$

$$\llcorner_{I} = \bigotimes_{i \in I} \llcorner_{i} = \begin{bmatrix} {}_{i_1}\llcorner \\ + \\ {}_{i_m}\llcorner \end{bmatrix}$$

$$I = \left\{ i_1 < \cdots < i_m \right\} \text{ geordnet}$$

$$\mathbb{K}_{\mathbb{N}} \triangledown \llcorner \in \mathbb{K} \triangle_{0}^{1:1}$$

$$\llcorner_{I} \wedge \llcorner'_{J} = \bigotimes_{i \in I} \llcorner_{i} \wedge \bigotimes_{j \in J} \llcorner'_{j} = \det \llcorner_{i} \wedge \llcorner'_{j} = \det \llcorner_{i}\, \eta \left(\overset{*}{\llcorner'_{j}} \right)$$

$$\mathbb{K}_{m} \triangledown \llcorner \to \mathbb{K} \triangledown \llcorner_{\triangle \overset{m}{\mathbb{K}}}$$

$$\bigotimes_{i \in I} \llcorner_{i} \,|\, \bigotimes_{j \in J} \urcorner^{j} = \det \llcorner_{i} \urcorner^{j}$$

$$\text{LHS} = m! \bigotimes_{i \in I} \llcorner_{i} \,|\, p_{-} \bigotimes_{j \in J} \urcorner^{j} = \sum_{I \xrightarrow{\asymp} \pi J} \overset{\pi}{-}1 \bigotimes_{i \in I} \llcorner_{i} \bigotimes_{i \in I} \urcorner^{\pi_i} = \sum_{\pi} \prod_{i \in I} \llcorner_{i} \urcorner^{\pi_i} = \text{RHS}$$

$$\mathbb{L}_{\mathbb{N}} \triangledown \llcorner = \mathbb{L} \otimes \mathbb{K}_{\mathbb{N}} \triangledown \llcorner$$

$$_{N}\llcorner \in \mathbb{K}_{n} \triangledown \llcorner$$

$$_{N}\llcorner \; \llcorner = \; _{N}\llcorner \det \llcorner$$