

$$\mathbb{K}^n \begin{smallmatrix} n \\ \triangleleft \mathbb{K} \end{smallmatrix} \leftarrow {}_n\mathbb{K}^n \ltimes \mathbb{K}^n \begin{smallmatrix} n \\ \triangleleft \mathbb{K} \end{smallmatrix}$$

$${}_n\mathbb{K}^n \rightarrow \left[\begin{smallmatrix} n+m+1 \\ m \end{smallmatrix} \right] \mathbb{K}^{\left[\begin{smallmatrix} n+m+1 \\ m \end{smallmatrix} \right]}$$

$$\mathbb{L} \mapsto \text{per } \bullet \mathbb{L}^\bullet$$

$$\mathbb{L} \cap \mathbb{L}^J = \sum_{|I|=m} \mathbb{L}^I \text{ per } {}_I\mathbb{L}^J$$

$$\mathbb{L} \sum_J \mathbb{L}^J \mathbb{J} = \sum_J \underbrace{\mathbb{L} \mathbb{L}^J}_J \mathbb{J} = \sum_{|I|=|J|} \mathbb{L}^I \overline{\sum_J \text{per } {}_I\mathbb{L}^J \mathbb{J}}$$

$$\mathbb{L} \underbrace{\mathbb{L}^\bullet \mathbb{J}} = \underbrace{\mathbb{L} \mathbb{L}^\bullet}_\mathbb{L} \mathbb{J}$$

$$\text{per } {}_I \underbrace{\mathbb{L} \mathbb{L}^\bullet}_I^K = \sum_{\pi} {}_{i_1} \mathbb{L} \mathbb{L}^{k_{\pi_1}} \dots {}_{i_m} \mathbb{L} \mathbb{L}^{k_{\pi_m}} = \sum_{j_1 \cdots j_m} \sum_{\pi} {}_{i_1} \mathbb{L}^{j_1} \mathbb{L}^{k_{\pi_1}} \dots {}_{i_m} \mathbb{L}^{j_m} \mathbb{L}^{k_{\pi_m}}$$

$$= \sum_{j_1 \cdots j_m \text{ dist}} \text{per} \left({}_{i_\mu} \mathbb{L}^{j_\mu} \mathbb{L}^{k_\nu} \right) = \sum_{j_1 \leqslant \cdots \leqslant j_m} \sum_{\sigma: \pi} {}_{i_1} \mathbb{L}^{j_{\sigma_1}} \mathbb{L}^{k_{\pi_1}} \dots {}_{i_m} \mathbb{L}^{j_{\sigma_m}} \mathbb{L}^{k_{\pi_m}}$$

$$= \sum_{j_1 \leqslant \cdots \leqslant j_m} \overline{\sum_{\sigma} {}_{i_1} \mathbb{L}^{j_{\sigma_1}} \dots {}_{i_m} \mathbb{L}^{j_{\sigma_m}}} \overline{\sum_{\tau} \mathbb{L}^{k_{\tau_1}} \dots \mathbb{L}^{k_{\tau_m}}} = \sum_J \text{per } {}_I\mathbb{L}^J \text{ per } {}_J\mathbb{L}^K$$

$$\tau=\pi\ltimes\sigma^{-1}$$

$${}_{j_{\sigma_i}}\mathbb{L}^{k_{\pi_i}}={}_{j_{\mu}}\mathbb{L}^{k_{\tau\mu}}\leftarrow\tau\left(\sigma_i\right)=\pi_i$$

$$\mathbb{L} \underbrace{\mathbb{L} \mathbf{x} \mathbb{L}^\bullet}_\mathbb{L} = \underbrace{\mathbb{L} \mathbb{L}^\bullet}_\mathbb{L} \mathbf{x} \underbrace{\mathbb{L} \mathbb{L}^\bullet}_\mathbb{L}$$

$$\underbrace{\mathbb{L} \mathbb{L}^{k_1}}_\mathbb{L} \mathbf{x} \dots \mathbf{x} \underbrace{\mathbb{L} \mathbb{L}^{k_m}}_\mathbb{L} = \sum_{j_1 \cdots j_m} \overbrace{\mathbb{L}^{j_1} \mathbb{L}^{k_1}}_{j_1} \mathbf{x} \dots \mathbf{x} \overbrace{\mathbb{L}^{j_m} \mathbb{L}^{k_m}}_{j_m}$$

$$= \sum_{|J|=m} \mathbb{L}^J \sum_{\pi} {}_{j_1} \mathbb{L}^{k_{\pi_1}} \dots {}_{j_m} \mathbb{L}^{k_{\pi_m}} = \sum_{|J|=m} \mathbb{L}^J \text{per } {}_J\mathbb{L}^K = \mathbb{L} \cap \mathbb{L}^K$$

$$\mathbb{K}^n \begin{smallmatrix} n \\ \triangleleft \mathbb{K} \end{smallmatrix} \stackrel{?}{\leftarrow} {}_n\mathbb{K}^n \ltimes \mathbb{K}^n \begin{smallmatrix} n \\ \triangleleft \mathbb{K} \end{smallmatrix}$$

$${}_n\mathbb{K}^n \rightarrow \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] \mathbb{K}^{\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]}$$

$$\mathbb{L} \mapsto \det \mathbb{L}$$

$$\mathbb{L} \cdot \mathbb{L}^J = \sum_{|I|=m} \mathbb{L}^I \det_I \mathbb{L}^J$$

$$\mathbb{L} \cdot \sum_J \mathbb{L}^J \cdot_J = \sum_J \underbrace{\mathbb{L} \cdot \mathbb{L}^J}_J = \sum_{|I|=|J|} \mathbb{L}^I \overline{\sum_J \det_I \mathbb{L}^J}$$

$$\mathbb{L} \cdot \underbrace{\mathbb{L} \cdot \mathbb{L}} = \underbrace{\mathbb{L} \cdot \mathbb{L}} \cdot \mathbb{L}$$

$$\det_I \underbrace{\mathbb{L} \cdot \mathbb{L}}^K = \sum_{\pi} -1_{i_1} \mathbb{L} \cdot \mathbb{L}^{k_{\pi_1}} \dots_{i_m} \mathbb{L} \cdot \mathbb{L}^{k_{\pi_m}} = \sum_{j_1 \dots j_m} \sum_{\pi} -1_{i_1} \mathbb{L}^{j_1} \cdot_{j_1} \mathbb{L}^{k_{\pi_1}} \dots_{i_m} \mathbb{L}^{j_m} \cdot_{j_m} \mathbb{L}^{k_{\pi_m}}$$

$$= \sum_{j_1 \dots j_m \text{ dist}} \det \left(\mathbb{L}^{j_{\mu}} \cdot_{j_{\mu}} \mathbb{L}^{k_{\nu}} \right) = \sum_{j_1 < \dots < j_m} \sum_{\sigma: \pi} -1_{i_1} \mathbb{L}^{j_{\sigma_1}} \cdot_{j_{\sigma_1}} \mathbb{L}^{k_{\pi_1}} \dots_{i_m} \mathbb{L}^{j_{\sigma_m}} \cdot_{j_{\sigma_m}} \mathbb{L}^{k_{\pi_m}}$$

$$= \sum_{j_1 < \dots < j_m} \overline{\sum_{\sigma} -1_{i_1} \mathbb{L}^{j_{\sigma_1}} \dots_{i_m} \mathbb{L}^{j_{\sigma_m}}} \overline{\sum_{\tau} -1_{j_1} \mathbb{L}^{k_{\tau_1}} \dots_{j_m} \mathbb{L}^{k_{\tau_m}}} = \sum_J \det_I \mathbb{L}^J \det_J \mathbb{L}^K$$

$$\tau = \pi \ltimes \sigma^{-1}$$

$$-1_{i_1} \dots -1_{i_m} = -1_{j_1} \dots -1_{j_m}$$

$$\mathbb{L}^{k_{\pi_i}}_{j_{\sigma_i}} = \mathbb{L}^{k_{\tau_i}}_{j_{\mu}} \Leftarrow \tau(\sigma_i) = \pi_i$$

$$\mathbb{L} \cdot \underbrace{\mathbb{L} \cdot \mathbb{L}} = \underbrace{\mathbb{L} \cdot \mathbb{L}} \cdot \mathbb{L}$$

$$\underbrace{\mathbb{L} \cdot \mathbb{L}^{k_1}} \cdot \dots \cdot \underbrace{\mathbb{L} \cdot \mathbb{L}^{k_m}} = \sum_{j_1 \dots j_m} \overbrace{\mathbb{L}^{j_1} \cdot_{j_1} \mathbb{L}^{k_1}} \cdot \dots \cdot \overbrace{\mathbb{L}^{j_m} \cdot_{j_m} \mathbb{L}^{k_m}}$$

$$= \sum_{|J|=m} \mathbb{L}^J \sum_{\pi} -1_{j_1} \mathbb{L}^{k_{\pi_1}} \dots_{j_m} \mathbb{L}^{k_{\pi_m}} = \sum_{|J|=m} \mathbb{L}^J \det_J \mathbb{L}^K = \mathbb{L} \cdot \mathbb{L}^K$$