$$\lfloor_{\triangle}\overset{m}{\mathbb{K}} \leftarrow \mathsf{C}|m \ltimes \lfloor_{\triangle}\overset{m}{\mathbb{K}}$$

$$\begin{bmatrix} \llcorner_1 \\ + \\ \llcorner_m \end{bmatrix} (\pi \ltimes \mathbb{1}) = \begin{bmatrix} \llcorner_{\pi 1} \\ + \\ \llcorner_{\pi m} \end{bmatrix} \mathbb{1}$$

$$\lfloor_{\triangle}\overset{\mathbb{N}}{\mathbb{1}} = \sum_m \lfloor_{\triangle}\overset{m}{\mathbb{1}}$$

$$\lfloor_{\triangle}\overset{m}{\mathbb{1}} = \frac{\lfloor \boxtimes \cdots \boxtimes \lfloor \xrightarrow{\mathbb{1}} \mathbb{1} \atop \text{m-lin}}{\pi \ltimes \mathbb{1} = -1^{\pi}\mathbb{1}} = \underset{m}{\mathbb{K}} \triangledown \lfloor_{\triangle}\mathbb{1} = \underbrace{\lfloor_{\triangle}\overset{m}{\mathbb{K}}} \boxtimes \mathbb{1}$$

$$\lfloor_{\triangle''}\overset{p+q}{\mathbb{1}} \overset{\boxtimes}{\longleftarrow} \lfloor_{\triangle}\overset{p}{\mathbb{1}} \boxtimes \lfloor_{\triangle'}\overset{q}{\mathbb{1}}$$

$$\lfloor_{\triangle}\overset{\mathbb{N}}{\mathbb{1}} = \sum_m \lfloor_{\triangle}\overset{m}{\mathbb{1}} \in \triangledown\overset{\mathbb{N}}{\mathbb{K}} \text{ Gel}$$

$$\lfloor_{\triangle}\overset{\mathbb{N}}{\mathbb{1}} \underset{p_{-}}{\overset{\sqsubset}{\rightleftarrows}} \lfloor_{\triangle}\overset{\mathbb{N}}{\mathbb{1}}$$

$$\lfloor_{\triangle}\overset{\mathbb{N}}{\mathbb{K}} = \sum_m \lfloor_{\triangle}\overset{m}{\mathbb{K}} \in \triangledown\overset{\mathbb{N}}{\mathbb{K}} \text{ Gel}$$

$$\lfloor_{\triangle}\overset{m}{\mathbb{K}} = \frac{\lfloor \boxtimes \cdots \boxtimes \lfloor \xrightarrow{\mathbb{1}} \mathbb{K} \atop \text{m-lin}}{\pi \ltimes \mathbb{1} = \overset{\pi}{-1}\mathbb{1}} \text{ symm}$$

$$\lfloor_{\triangle}\overset{p+q}{\mathbb{K}} \overset{\boxtimes}{\longleftarrow} \lfloor_{\triangle}\overset{p}{\mathbb{K}} \boxtimes \lfloor_{\triangle}\overset{q}{\mathbb{K}}$$

$${}_{M}\llcorner \left(\overset{p}{\mathbb{1}} \boxtimes \mathbb{1}'\right) = \sum_{P \subset M}^{\sharp P = p} \overset{\overline{P > M \llcorner P}}{-1} \overbrace{{}_{P}\llcorner \mathbb{1}} \overbrace{{}_{M \llcorner P}\llcorner \mathbb{1}'}$$

$$\begin{bmatrix} {}_1\llcorner \\ + \\ {}_m\llcorner \end{bmatrix} \mathbb{1} \boxtimes \mathbb{1}' = \sum_{1 \leqslant \nu_1 \leqslant \cdots \leqslant \nu_p \leqslant m} \overset{\nu}{-1} \begin{bmatrix} {}_{\nu_1}\llcorner \\ + \\ {}_{\nu_p}\llcorner \end{bmatrix} \mathbb{1} \begin{bmatrix} {}_{\nu_{p+1}}\llcorner \\ + \\ {}_{\nu_m}\llcorner \end{bmatrix} \mathbb{1}'$$

$$\overset{\nu}{-1} = \overset{\overline{P > M \llcorner P}}{-1}$$

$$(\bowtie \mathbb{L}) \bowtie \left(\bowtie \mathbb{L}'\right) = \mathbb{L} \bowtie \mathbb{L}'$$

$$(\bowtie \mathbb{L}) \bowtie \mathbb{1} = \mathbb{L}\mathbb{1}$$

$$\blacktriangledown_{I} \llcorner = \bowtie_{i \in I} (\blacktriangledown_{i} \llcorner)$$

$$(\blacktriangledown_{I} \llcorner) \blacktriangledown \left(\blacktriangledown_{J} \llcorner'\right) = {}_{I}\llcorner \blacktriangledown {}_{J}\llcorner'$$

$$\urcorner^{I} \blacktriangledown \urcorner'^{J} = \left(\bowtie_{i \in I} \urcorner^{i}\right) \blacktriangledown \left(\bowtie_{j \in J} \urcorner'^{j}\right) = \text{per} \left(\urcorner^{i} \blacktriangledown \urcorner'^{j}\right)$$

$$\left(\bowtie_{i \in I} \llcorner_{i}\right) \blacktriangledown \left(\bowtie_{j \in J} \llcorner'_{j}\right) = \text{per} \left(\llcorner_{i} \blacktriangledown \llcorner'_{j}\right) = \text{per} \left(\llcorner_{i} \left(\blacktriangledown \llcorner'_{j}\right)\right) = \left(\bowtie_{i \in I} \llcorner_{i}\right) \left(\bowtie_{j \in J} \left(\blacktriangledown \llcorner'_{j}\right)\right)$$

$$(\bowtie(\blacktriangledown \llcorner_{i})) \blacktriangledown \left(\bowtie \urcorner^{j}\right) = (\blacktriangledown (\bowtie \llcorner_{i})) \blacktriangledown \left(\bowtie \urcorner^{j}\right) = (\bowtie \llcorner_{i}) \left(\bowtie \urcorner^{j}\right) = \text{per} \left(\llcorner_{i} \urcorner^{j}\right) = \text{per} \left((\blacktriangledown \llcorner_{i}) \blacktriangledown \urcorner^{j}\right)$$

$$= \mathbb{K}_{m} \triangledown \llcorner \triangle \mathbb{K} \overset{?}{\leftarrow} \llcorner \triangle \mathbb{K}^{m}$$

$$\bowtie_{i \in I} \llcorner_{i} | \bowtie_{j \in J} \urcorner^{j} = \text{per} \llcorner_{i} \urcorner^{j}$$

$$\text{LHS} = m! \bowtie_{i \in I} \llcorner_{i} | p_{-} \bowtie_{j \in J} \urcorner^{j} = \sum_{I \xrightarrow{\asymp} \pi J} \overset{\pi}{-}1 \bowtie_{i \in I} \llcorner_{i} \bowtie_{i \in I} \urcorner^{\pi_i} = \sum_{\pi} \prod_{i \in I} \llcorner_{i} \urcorner^{\pi_i} = \text{RHS}$$

$$\llcorner \triangle \mathbb{K}^{\mathbb{N}} \underset{p_{-}}{\overset{\sqsubset}{\rightleftarrows}} \llcorner \triangle \mathbb{K}^{\mathbb{N}}$$

$$p_{-} \urcorner = \frac{1}{m!} \sum_{\pi \in \mathsf{C}|m} \overset{\pi}{-}1 \, \pi \ltimes \urcorner$$

$$p_{-} (\tau \ltimes \urcorner) = \overset{\tau}{-}1 \, p_{-} \urcorner$$

$$p_{-} \left(p_{-} \urcorner\right) = p_{-} \urcorner$$

$$\left(p_{-} \urcorner\right) \bowtie \left(p_{-} \urcorner'\right) = \begin{bmatrix} m \\ pq \end{bmatrix} p_{-} \left(\urcorner \bowtie \urcorner'\right)$$

$$m! \, {}_{M}\llcorner \, p_{-} \left(\urcorner \bowtie \urcorner'\right) = \sum_{\pi \in \mathsf{C}|M} \overset{\pi}{-}1 \, {}_{\pi_M}\llcorner \left(\urcorner \bowtie \urcorner'\right) = \sum_{\pi \in \mathsf{C}|M} \overset{\pi}{-}1 \left({}_{\pi_{M_<}}\llcorner \urcorner\right) \times \left({}_{\pi_{M_>}}\llcorner \urcorner'\right)$$

$$= \sum_{|P| = p} \overset{\overline{P > M \llcorner P}}{-1} \sum_{M_< \xrightarrow{\asymp} \sigma P} \overset{\sigma}{-}1 \left({}_{\sigma}\llcorner \urcorner\right) \times \sum_{M_> \xrightarrow{\asymp} \tau M \llcorner P} \overset{\tau}{-}1 \left({}_{\tau}\llcorner \urcorner'\right) = \sum_{|P| = p} \overset{\overline{P > M \llcorner P}}{-1} p! \left({}_{P}\llcorner p_{-} \urcorner\right) \times q! \left({}_{M \llcorner P}\llcorner p_{-} \urcorner'\right)$$

$$\Leftarrow \overset{\pi}{-}1 = \prod_{M_< \ni i < j \in M_<} \pi_i \sharp \pi_j \prod_{M_> \ni i < j \in M_>} \pi_i \sharp \pi_j \prod_{M_< \ni i | j \in M_>} \pi_i \sharp \pi_j = \overset{\sigma}{-}1 \overset{\tau}{-}1 \overset{\overline{P > M \llcorner P}}{-1} | \sigma = \pi_{M_<} | \tau = \pi_{M_>}$$

$$p_{-} \left(\left(p_{-} \urcorner\right) \bowtie \urcorner'\right) = p_{-} \left(\urcorner \bowtie \urcorner'\right) = p_{-} \urcorner \bowtie \left(p_{-} \urcorner'\right)$$

$$\begin{bmatrix} m \\ pq \end{bmatrix} p_- \left(\left(p_- \mathbb{1}\right) \bowtie \mathbb{1}'\right) = p_- \left(\left(p_- \mathbb{1}\right) \bowtie \left(p_- \mathbb{1}'\right)\right) = \left(p_- \mathbb{1}\right) \bowtie \left(p_- \mathbb{1}'\right) = \begin{bmatrix} m \\ pq \end{bmatrix} p_- \mathbb{1} \bowtie \left(p_- \mathbb{1}'\right)$$

$$\mathsf{L}_{\triangle}{}^{\mathbb{N}}\mathbb{K} \text{ assoc super-comm}$$

$$\left(\mathbb{1}\bowtie\mathbb{1}'\right)\bowtie\mathbb{1}'' = \begin{bmatrix} m \\ p+q{:}r \end{bmatrix} p_- \left(\left(\mathbb{1}\bowtie\mathbb{1}'\right)\bowtie\mathbb{1}''\right) = \begin{bmatrix} m \\ p{:}q{:}r \end{bmatrix} p_- \, p_- \left(\mathbb{1}\bowtie\mathbb{1}'\right)\bowtie\mathbb{1}'' = \begin{bmatrix} m \\ p{:}q{:}r \end{bmatrix} p_- \left(\mathbb{1}\bowtie\mathbb{1}'\bowtie\mathbb{1}''\right)$$

$$\mathbb{1}\bowtie\mathbb{1}' = \begin{bmatrix} m \\ pq \end{bmatrix} p_- \left(\mathbb{1}\bowtie\mathbb{1}'\right) = \begin{bmatrix} m \\ pq \end{bmatrix} p_- \tau \ltimes \left(\mathbb{1}'\bowtie\mathbb{1}\right) = \overset{\tau}{-}1 \left(\mathbb{1}'\bowtie\mathbb{1}\right)$$

$$\Leftarrow \overset{\tau}{-}1 = \prod_{M_< \ni i<j \in M_<} \frac{\tau_i \sharp \tau_j}{8} (=1) \prod_{M_> \ni i<j \in M_>} \frac{\tau_i \sharp \tau_j}{8} (=1) \prod_{M_< \ni i:j \in M_>} \frac{\tau_i \sharp \tau_j}{8} (=-1) = \overset{pq}{-}1$$

$$\tau \begin{bmatrix} M_< \\ M_> \end{bmatrix} = \begin{bmatrix} M_> \\ M_< \end{bmatrix}$$

$$\mathbb{1}^0 \bowtie \cdots \bowtie \mathbb{1}^k = \begin{bmatrix} m \\ p_0 \cdots p_k \end{bmatrix} p_- \mathbb{1}^0 \bowtie \cdots \bowtie \mathbb{1}^k$$

$$\text{LHS} = \mathbb{1}^0 \bowtie \left(\mathbb{1}^1 \bowtie \cdots \bowtie \mathbb{1}^k\right) = \begin{bmatrix} m \\ p_0{:}p_1 \cdots p_k \end{bmatrix} p_- \mathbb{1}^0 \bowtie \left(\mathbb{1}^1 \bowtie \cdots \bowtie \mathbb{1}^k\right) = \begin{bmatrix} m \\ p_0{:}p_1 \cdots p_k \end{bmatrix} \begin{bmatrix} p_1 + \cdots + p_k \\ p_1 \cdots p_k \end{bmatrix} p_- \mathbb{1}^0 \bowtie \left(p_- \mathbb{1}^1 \bowtie \cdots \bowtie \mathbb{1}^k\right) = \text{R}$$

$$\mathsf{L}_{\triangle}{}^{m}\mathbb{K} = \mathsf{L}_{\triangle}\mathbb{K}{}_{\triangle}{}^{m}\mathbb{K} \ni \mathbb{1}^J = \bowtie_{j \in J} \mathbb{1}^j$$

$$\begin{tikzcd}
\mathsf{L}_{\triangle}{}^{m}\mathbb{K} \arrow[r, shift left, "a"] \arrow[d, shift right, "\ast^m"'] & \mathbb{K}_m \triangledown \bar{\mathsf{L}} \arrow[l, shift left, "\bowtie"] \arrow[r, shift left, "c"] \arrow[d, shift right, "g"'] & \uplus | \bar{\mathsf{L}}_{\triangle}{}^{\mathbb{N}}\mathbb{K} \arrow[l, shift left, "d"] \arrow[d, shift right, "i"'] \\
\bar{\mathsf{L}}_{\triangle}{}^{n-m}\mathbb{K} \arrow[u, shift right, "f"'] \arrow[r, shift left, "k"] & \mathbb{K}_{n-m} \triangledown \mathsf{L} \arrow[u, shift right, "h"'] \arrow[l, shift left, "\bowtie"] \arrow[r, shift left, "m"] & \uplus | \mathbb{K}_{\mathbb{N}} \triangledown \mathsf{L} \arrow[u, shift right, "j"'] \arrow[l, shift left, "\mathbb{L}^N \bowtie"]
\end{tikzcd}$$

$$\ast\left(\bowtie \mathbb{L}^m\right) = \mathbb{L}^{\,n-m} \vDash \mathbb{L}^N$$

$$\mathbb{1}' \bowtie (\ast\mathbb{1}) = \underbrace{\mathbb{1}\bowtie\mathbb{1}'}\, \mathbb{L}^N$$

$$\mathbb{1}\bowtie\mathbb{1}' = {}_N\mathbb{L} \overbrace{\mathbb{1}'\bowtie(\ast\mathbb{1})}$$

$$\mathbb{1}'\bowtie\overbrace{\ast\underbrace{\bowtie\mathbb{L}}} = \mathbb{1}'\bowtie\underbrace{\mathbb{L} \vDash \mathbb{L}^N} = \underbrace{\mathbb{L}\mathbb{1}'}\,\mathbb{L}^N = \overbrace{\underbrace{\bowtie\mathbb{L}}\bowtie\mathbb{1}'}\,\mathbb{L}^N$$

$$\underbrace{\mathbb{L}'} \mathbb{\Xi} \overbrace{\ast \underbrace{\mathbb{L}}} = \underbrace{\mathbb{L}'} \mathbb{\Xi} \underbrace{\mathbb{L} \vDash \mathbb{T}^{N}} = \underbrace{\mathbb{L}\,\mathbb{L}'} \mathbb{T}^{N} = \overbrace{\underbrace{\mathbb{L}} \underbrace{\mathbb{L}'}} \mathbb{T}^{N}$$

$$\mathbb{L} \vDash \mathbb{T}^{N} = \overbrace{\underbrace{\mathbb{T}^{N}} \dashv \underbrace{\mathbb{L}}}$$

$$\ast \mathbb{T} = \overbrace{\underbrace{\mathbb{T}^{N}} \dashv \mathbb{T}}$$

$$\mathbb{L}' \underbrace{\mathbb{L} \vDash \mathbb{T}^{N}} = \underbrace{\mathbb{L}' \mathbb{\Xi} \mathbb{L}} \mathbb{T}^{N} = \underbrace{\mathbb{L}' \mathbb{\Xi} \mathbb{L}} \, \mathbb{T}^{N} = \mathbb{L}' \overbrace{\underbrace{\mathbb{T}^{N}} \dashv \underbrace{\mathbb{L}}}$$

$$\underbrace{{}_{N}\mathbb{L} \dashv \overset{m}{\mathbb{T}}} \vDash \mathbb{T}^{N} = \overset{m\,(n-m)}{-1} \mathbb{T} \underbrace{{}_{N}\mathbb{L}\,\mathbb{L}^{N}}$$

$$\overset{m\,(n-m)}{-1} \mathbb{L} \overbrace{\underbrace{{}_{N}\mathbb{L} \dashv \mathbb{T}} \vDash \mathbb{T}^{N}} = \overset{m\,(n-m)}{-1} \overbrace{\mathbb{L}\mathbb{\Xi} \underbrace{{}_{N}\mathbb{L} \dashv \mathbb{T}}} \mathbb{T}^{N} = \overbrace{\underbrace{{}_{N}\mathbb{L} \dashv \mathbb{T}} \mathbb{\Xi}\mathbb{L}} \mathbb{T}^{N} = \overbrace{{}_{N}\mathbb{L} \dashv \mathbb{T}} \underbrace{\mathbb{L} \vDash \mathbb{T}^{N}} = \underbrace{\mathbb{L}\mathbb{T}} \, {}_{N}\mathbb{L}\, \mathbb{T}^{N}$$

$$\overset{n-m}{\ast} \overset{m}{\ast} = \overset{m\,(n-m)}{-1} \mathbb{T}^{N} \, \mathbb{T}^{N}$$

$$\overset{n-m}{\ast} \overbrace{\overset{m}{\ast} \underbrace{\mathbb{L}}} = \overset{n-m}{\ast} \underbrace{\mathbb{L} \vDash \mathbb{T}^{N}} = \overset{n-m}{\ast} \overbrace{\underbrace{\mathbb{T}^{N}} \dashv \underbrace{\mathbb{L}}} = \overbrace{\underbrace{\mathbb{T}^{N}} \dashv \underbrace{\mathbb{L}}} \vDash \mathbb{T}^{N} = \overset{m\,(n-m)}{-1} \underbrace{\mathbb{L}} \, \mathbb{T}^{N} \, \mathbb{T}^{N}$$

$$\mathbb{L}\overset{\mathbb{N}}{\mathbb{K}} = \sum_{m} \mathbb{L}\overset{m}{\mathbb{K}} \in \overset{\mathbb{N}}{\mathbb{K}} \ \mathrm{Gel}$$

$$\mathbb{L}\overset{m}{\mathbb{K}} = \frac{\mathbb{L}\,\mathbb{\Xi}\cdots\mathbb{\Xi}\,\mathbb{L} \xrightarrow{\ \mathbb{T}\ } \mathbb{K} \quad \text{m-lin}}{\pi \ltimes \mathbb{T} = \overset{\pi}{-1}\mathbb{T}} \ \text{anti-symm}$$

$$\mathbb{L}\overset{p+q}{\mathbb{K}} \xleftarrow{\ \mathbb{\Xi}\ } \mathbb{L}\overset{p}{\mathbb{K}} \,\mathbb{\Xi}\, \mathbb{L}\overset{q}{\mathbb{K}}$$

$$_{M}\mathbb{L} \left(\overset{p}{\mathbb{T}} \mathbb{\Xi} \mathbb{T}' \right) = \sum_{\substack{\sharp P = p \\ P \subset M}} \overset{\overline{P > M \llcorner P}}{-1} \, \overbrace{{}_{P}\mathbb{L}\,\mathbb{T}} \, \overbrace{{}_{M \llcorner P}\mathbb{L}\,\mathbb{T}'}$$

$$\begin{bmatrix} {}_{1}\mathbb{L} \\ + \\ {}_{m}\mathbb{L} \end{bmatrix} \mathbb{T}\mathbb{\Xi}\mathbb{T}' = \sum_{1 \leqslant \nu_1 < \cdots < \nu_p \leqslant m} \overset{\nu}{-1} \begin{bmatrix} {}_{\nu_1}\mathbb{L} \\ + \\ {}_{\nu_p}\mathbb{L} \end{bmatrix} \mathbb{T} \begin{bmatrix} {}_{\nu_{p+1}}\mathbb{L} \\ + \\ {}_{\nu_m}\mathbb{L} \end{bmatrix} \mathbb{T}'$$

$$\overset{\nu}{-1} = \overset{\overline{P > M \llcorner P}}{-1}$$

$$(\mathbb{L}) \left(\mathbb{L}' \right) := \mathbb{L} \, \mathbb{L}'$$

$$(\mathbb{L}) \, \mathbb{T} = \mathbb{L}\,\mathbb{T}$$

$$ {}_{I}\mathbb{L} = \underset{i \in I}{\mathbb{\Xi}} ({}_{i}\mathbb{L})$$

$$({}_{I}\mathbb{L}) \left({}_{J}\mathbb{L}' \right) = {}_{I}\mathbb{L} \ {}_{J}\mathbb{L}'$$

$$\mathbb{T}^{I} \, \mathbb{T}'^{J} = \left(\underset{i \in I}{\mathbb{\Xi}} \mathbb{T}^{i} \right) \left(\underset{j \in J}{\mathbb{\Xi}} \mathbb{T}'^{j} \right) = \det \left(\mathbb{T}^{i} \, \mathbb{T}'^{j} \right)$$

$$\left(\bigwedge_{i\in I}\llcorner_i\right)\wedge\left(\bigwedge_{j\in J}\llcorner'_j\right)=\det\left(\llcorner_i\wedge\llcorner'_j\right)=\det\left(\llcorner_i\left(\wedge\llcorner'_j\right)\right)=\left(\bigwedge_{i\in I}\llcorner_i\right)\left(\bigwedge_{j\in J}\left(\wedge\llcorner'_j\right)\right)$$

$$(\bigwedge(\wedge\llcorner_i))\wedge\left(\bigwedge\urcorner^j\right)=(\wedge(\bigwedge\llcorner_i))\wedge\left(\bigwedge\urcorner^j\right)=(\bigwedge\llcorner_i)\left(\bigwedge\urcorner^j\right)=\det\left(\llcorner_i\urcorner^j\right)=\det\left((\wedge\llcorner_i)\wedge\urcorner^j\right)$$

$$=\ \mathbb{K}_m\urcorner\llcorner_{\triangle}\mathbb{K}\xleftarrow{?}\llcorner_{\triangle}\mathbb{K}^m$$

$$\bigwedge_{i\in I}\llcorner_i\,|\bigwedge_{j\in J}\urcorner^j=\det\llcorner_i\urcorner^j$$

$$\text{LHS }=m!\bigwedge_{i\in I}\llcorner_i\,|\,p_{-}\bigwedge_{j\in J}\urcorner^j\ =\sum_{I\xrightarrow{\asymp}\pi J}\overset{\pi}{-1}\bigwedge_{i\in I}\llcorner_i\bigwedge_{i\in I}\urcorner^{\pi_i}=\sum_{\pi}\prod_{i\in I}\llcorner_i\urcorner^{\pi_i}\ =\ \text{RHS}$$

$$\llcorner_{\triangle}\mathbb{K}^{\mathbb{N}}\ \underset{p_{-}}{\overset{\sqsubset}{\rightleftarrows}}\ \llcorner_{\triangle}\mathbb{K}^{\mathbb{N}}$$

$$p_{-}\urcorner=\frac{1}{m!}\sum_{\pi\in\mathsf{C}|m}\overset{\pi}{-1}\,\pi\ltimes\urcorner$$

$$p_{-}\left(\tau\ltimes\urcorner\right)=\overset{\tau}{-1}\,p_{-}\urcorner$$

$$p_{-}\left(p_{-}\urcorner\right)=p_{-}\urcorner$$

$$\left(p_{-}\urcorner\right)\bigwedge\left(p_{-}\urcorner'\right)=\begin{bmatrix}m\\pq\end{bmatrix}p_{-}\left(\urcorner\bigwedge\urcorner'\right)$$

$$m!\,{}_M\llcorner\,p_{-}\left(\urcorner\bigwedge\urcorner'\right)=\sum_{\pi\in\mathsf{C}|M}\overset{\pi}{-1}\,{}_{\pi_M}\llcorner\left(\urcorner\bigwedge\urcorner'\right)=\sum_{\pi\in\mathsf{C}|M}\overset{\pi}{-1}\left({}_{\pi_{M_<}}\llcorner\urcorner\right)\times\left({}_{\pi_{M_>}}\llcorner\urcorner'\right)$$

$$=\sum_{|P|=p}\overline{\overset{P>M\llcorner P}{-1}}\sum_{M_<\xrightarrow{\asymp}\sigma P}\overset{\sigma}{-1}\left({}_\sigma\llcorner\urcorner\right)\times\sum_{M_>\xrightarrow{\asymp}\tau M\llcorner P}\overset{\tau}{-1}\left({}_\tau\llcorner\urcorner'\right)=\sum_{|P|=p}\overline{\overset{P>M\llcorner P}{-1}}\,p!\left({}_P\llcorner p_{-}\urcorner\right)\times q!\left({}_{M\llcorner P}\llcorner p_{-}\urcorner'\right)$$

$$\Leftarrow\overset{\pi}{-1}=\prod_{M_<\ni i<j\in M_<}\pi_i\sharp\pi_j\prod_{M_>\ni i<j\in M_>}\pi_i\sharp\pi_j\prod_{M_<\ni i:j\in M_>}\pi_i\sharp\pi_j=\overset{\sigma}{-1}\,\overset{\tau}{-1}\,\overline{\overset{P>M\llcorner P}{-1}}\,|\sigma=\pi_{M_<}|\tau=\pi_{M_>}$$

$$p_{-}\left(\left(p_{-}\urcorner\right)\bigwedge\urcorner'\right)=p_{-}\left(\urcorner\bigwedge\urcorner'\right)=p_{-}\urcorner\bigwedge\left(p_{-}\urcorner'\right)$$

$$\begin{bmatrix}m\\pq\end{bmatrix}p_{-}\left(\left(p_{-}\urcorner\right)\bigwedge\urcorner'\right)=p_{-}\left(\left(p_{-}\urcorner\right)\bigwedge\left(p_{-}\urcorner'\right)\right)=\left(p_{-}\urcorner\right)\bigwedge\left(p_{-}\urcorner'\right)=\begin{bmatrix}m\\pq\end{bmatrix}p_{-}\urcorner\bigwedge\left(p_{-}\urcorner'\right)$$

$$\llcorner_{\triangle}\mathbb{K}^{\mathbb{N}}\ \text{assoc super-comm}$$

$$\left(\urcorner\bigwedge\urcorner'\right)\bigwedge\urcorner''=\begin{bmatrix}m\\p+q{:}r\end{bmatrix}p_{-}\left(\left(\urcorner\bigwedge\urcorner'\right)\bigwedge\urcorner''\right)=\begin{bmatrix}m\\p{:}q{:}r\end{bmatrix}p_{-}\,p_{-}\left(\urcorner\bigwedge\urcorner'\right)\bigwedge\urcorner''=\begin{bmatrix}m\\p{:}q{:}r\end{bmatrix}p_{-}\left(\urcorner\bigwedge\urcorner'\bigwedge\urcorner''\right)$$

$$\mathbb{T}\boxtimes\mathbb{T}' = \begin{bmatrix} m \\ pq \end{bmatrix} p_- \left(\mathbb{T}\boxtimes\mathbb{T}'\right) = \begin{bmatrix} m \\ pq \end{bmatrix} p_- \tau \ltimes \left(\mathbb{T}'\boxtimes\mathbb{T}\right) = \overset{\tau}{-}1 \left(\mathbb{T}'\boxtimes\mathbb{T}\right)$$

$$\Leftarrow \overset{\tau}{-}1 = \prod_{M_< \ni i<j \in M_<} \frac{\tau_i \sharp \tau_j}{8} (=1) \prod_{M_> \ni i<j \in M_>} \frac{\tau_i \sharp \tau_j}{8} (=1) \prod_{M_< \ni i:j \in M_>} \frac{\tau_i \sharp \tau_j}{8} (=-1) = \overset{pq}{-}1$$

$$\tau \begin{bmatrix} M_< \\ M_> \end{bmatrix} = \begin{bmatrix} M_> \\ M_< \end{bmatrix}$$

$$\mathbb{T}^0 \boxtimes \cdots \boxtimes \mathbb{T}^k = \begin{bmatrix} m \\ p_0 \cdots p_k \end{bmatrix} p_- \mathbb{T}^0 \boxtimes \cdots \boxtimes \mathbb{T}^k$$

$$\text{LHS} = \mathbb{T}^0 \boxtimes \left(\mathbb{T}^1 \boxtimes \cdots \boxtimes \mathbb{T}^k\right) = \begin{bmatrix} m \\ p_0 : p_1 \cdots p_k \end{bmatrix} p_- \mathbb{T}^0 \boxtimes \left(\mathbb{T}^1 \boxtimes \cdots \boxtimes \mathbb{T}^k\right) = \begin{bmatrix} m \\ p_0 : p_1 \cdots p_k \end{bmatrix} \begin{bmatrix} p_1 + \cdots + p_k \\ p_1 \cdots p_k \end{bmatrix} p_- \mathbb{T}^0 \boxtimes \left(p_- \mathbb{T}^1 \boxtimes \cdots \boxtimes \mathbb{T}^k\right) = \text{R}$$

$$\mathbb{L}_{\triangle} \overset{m}{\mathbb{K}} = \mathbb{L}_{\triangle} \mathbb{K} \triangle \overset{m}{\mathbb{K}} \ni \mathbb{T}^J = \boxtimes_{j \in J} \mathbb{T}^j$$

$$\begin{tikzcd}
\mathbb{L}_{\triangle} \overset{m}{\mathbb{K}} \arrow[r, shift left, "a"] \arrow[d, shift right, "\overset{m}{\divideontimes}"'] & \mathbb{K}_m \triangleleft \bar{\mathbb{L}} \arrow[l, shift left, "\bowtie"] \arrow[r, shift left, "c"] \arrow[d, shift right, "g"'] & \Psi | \bar{\mathbb{L}}_{\triangle} \overset{\mathbb{N}}{\mathbb{K}} \arrow[l, shift left, "d"] \arrow[d, shift right, "i"'] \\
\bar{\mathbb{L}}_{\triangle} \overset{n-m}{\mathbb{K}} \arrow[u, shift right, "f"'] \arrow[r, shift left, "k"] & \mathbb{K}_{n-m} \triangleleft \mathbb{L} \arrow[u, shift right, "h"'] \arrow[l, shift left, "\bowtie"] \arrow[r, shift left, "m"] & \Psi | \mathbb{K}_{\mathbb{N}} \triangleleft \mathbb{L} \arrow[u, shift right, "j"'] \arrow[l, shift left, "\mathbb{T}^N \bowtie"]
\end{tikzcd}$$

$$\divideontimes\left(\bowtie \overset{m}{\mathbb{L}}\right) = \mathbb{L} \overset{n-m}{\vDash} \mathbb{T}^N$$

$$\mathbb{T}' \boxtimes (\divideontimes \mathbb{T}) = \underbrace{\mathbb{T} \bowtie \mathbb{T}'} \mathbb{T}^N$$

$$\mathbb{T} \bowtie \mathbb{T}' = {}_N \mathbb{T} \overbrace{\mathbb{T}' \boxtimes (\divideontimes \mathbb{T})}$$

$$\mathbb{T}' \boxtimes \overbrace{\divideontimes \underbrace{\bowtie \mathbb{L}}} = \mathbb{T}' \boxtimes \underbrace{\mathbb{L} \vDash \mathbb{T}^N} = \underbrace{\mathbb{L} \mathbb{T}'} \mathbb{T}^N = \overbrace{\underbrace{\bowtie \mathbb{L}} \bowtie \mathbb{T}'} \mathbb{T}^N$$

$$\underbrace{\bowtie \mathbb{L}'} \boxtimes \overbrace{\divideontimes \underbrace{\bowtie \mathbb{L}}} = \underbrace{\bowtie \mathbb{L}'} \boxtimes \underbrace{\mathbb{L} \vDash \mathbb{T}^N} = \underbrace{\mathbb{L} \bowtie \mathbb{L}'} \mathbb{T}^N = \overbrace{\underbrace{\bowtie \mathbb{L}} \bowtie \underbrace{\bowtie \mathbb{L}'}} \mathbb{T}^N$$

$$\mathbb{L} \vDash \mathbb{T}^N = \bowtie \overbrace{\underbrace{\mathbb{T}^N \bowtie} \dashv \overbrace{\bowtie \mathbb{L}}}$$

$$\divideontimes \mathbb{T} = \bowtie \overbrace{\underbrace{\mathbb{T}^N \bowtie} \dashv \mathbb{T}}$$

$$\mathbb{L}' \underbrace{\mathbb{L} \vDash \mathbb{T}^{N}} = \underbrace{\mathbb{L}' \boxtimes \mathbb{L}}\, \mathbb{T}^{N} = \underbrace{\mathbb{L}' \boxtimes \mathbb{L}}\, \blacktriangle\, \underbrace{\mathbb{T}^{N} \blacktriangledown} = \mathbb{L}' \blacktriangle\, \overbrace{\underbrace{\mathbb{T}^{N} \blacktriangledown} \dashv \underbrace{\blacktriangle \mathbb{L}}}$$

$$\underbrace{{}_{N}\mathbb{T} \dashv \overset{m}{\rrbracket}} \vDash \mathbb{T}^{N} = {-1}^{m\,(n-m)}\, \rrbracket\, \underbrace{{}_{N}\mathbb{T}\, \mathbb{T}^{N}}$$

$${-1}^{m\,(n-m)}\, \mathbb{L}\, \overbrace{\underbrace{{}_{N}\mathbb{T} \dashv \rrbracket} \vDash \mathbb{T}^{N}} = {-1}^{m\,(n-m)}\, \overbrace{\mathbb{L} \boxtimes \underbrace{{}_{N}\mathbb{T} \dashv \rrbracket}}\, \mathbb{T}^{N} = \overbrace{\underbrace{{}_{N}\mathbb{T} \dashv \rrbracket} \boxtimes \mathbb{L}}\, \mathbb{T}^{N} = \overbrace{{}_{N}\mathbb{T} \dashv \rrbracket}\, \underbrace{\mathbb{L} \vDash \mathbb{T}^{N}} = \underbrace{\mathbb{L} \rrbracket}\, {}_{N}\mathbb{T}\, \mathbb{T}^{N}$$

$$\overset{n-m}{\ast}\; \overset{m}{\ast} = {-1}^{m\,(n-m)}\, \mathbb{T}^{N} \blacktriangledown \mathbb{T}^{N}$$

$$\overset{n-m}{\ast}\; \overbrace{\overset{m}{\ast} \underbrace{\blacktriangle \mathbb{L}}} = \overset{n-m}{\ast}\; \underbrace{\mathbb{L} \vDash \mathbb{T}^{N}} = \overset{n-m}{\ast}\, \blacktriangle\, \overbrace{\underbrace{\mathbb{T}^{N} \blacktriangledown} \dashv \underbrace{\blacktriangle \mathbb{L}}} = \overbrace{\underbrace{\mathbb{T}^{N} \blacktriangledown} \dashv \underbrace{\blacktriangle \mathbb{L}}} \vDash \mathbb{T}^{N} = {-1}^{m\,(n-m)}\, \underbrace{\blacktriangle \mathbb{L}}\, \mathbb{T}^{N} \blacktriangledown \mathbb{T}^{N}$$