

$$\mathbb{L}_{\triangleleft \mathbb{K}}^m \leftarrow \mathbb{C}|m \ltimes \mathbb{L}_{\triangleleft \mathbb{K}}^m$$

$$\begin{bmatrix} \mathbb{L}_1 \\ + \\ \mathbb{L}_m \end{bmatrix} (\pi \ltimes \mathbb{T}) = \begin{bmatrix} \mathbb{L}_{\pi 1} \\ + \\ \mathbb{L}_{\pi m} \end{bmatrix} \mathbb{T}$$

$$\mathbb{L}_{\triangleleft \mathbb{I}}^{\mathbb{N}} = \sum_m \mathbb{L}_{\triangleleft \mathbb{I}}^m$$

$$\mathbb{L}_{\triangleleft \mathbb{I}}^m = \frac{\mathbb{L} \mathbf{x} \cdots \mathbf{x} \mathbb{L} \xrightarrow{\mathbb{T}} \mathbb{I}}{\pi \ltimes \mathbb{T} = -1^{\pi} \mathbb{T}} = \frac{\mathbb{K}_m \nabla \mathbb{L}_{\triangleleft \mathbb{I}}}{\pi \ltimes \mathbb{T} = -1^{\pi} \mathbb{T}} = \underbrace{\mathbb{L}_{\triangleleft \mathbb{K}}^m}_{\mathbb{K}} \mathbf{x} \mathbb{I}$$

$$\mathbb{L}_{\triangleleft \mathbb{I}}^{\textcolor{teal}{p}+q} \mathbf{x} \mathbb{L}_{\triangleleft \mathbb{I}}^p \mathbf{x} \mathbb{L}_{\triangleleft \mathbb{I}}^{\textcolor{teal}{q}}$$

$$\mathbb{L}_{\triangleleft \mathbb{I}}^{\mathbb{N}} = \sum_m \mathbb{L}_{\triangleleft \mathbb{I}}^m \in \nabla \mathbb{K} \text{ Gel}$$

$$\mathbb{L}_{\triangleleft \mathbb{I}}^{\mathbb{N}} \overset{\mathbb{C}}{\underset{p_-}{\rightleftarrows}} \mathbb{L}_{\triangleleft \mathbb{I}}^{\mathbb{N}}$$

$$\mathbb{L}_{\triangleleft \mathbb{K}}^{\mathbb{N}} = \sum_m \mathbb{L}_{\triangleleft \mathbb{K}}^m \in \nabla \mathbb{K} \text{ Gel}$$

$$\mathbb{L}_{\triangleleft \mathbb{K}}^m = \frac{\mathbb{L} \mathbf{x} \cdots \mathbf{x} \mathbb{L} \xrightarrow{\mathbb{T}} \mathbb{K}}{\pi \ltimes \mathbb{T} = -1^{\pi} \mathbb{T}} \text{ symm}$$

$$\mathbb{L}_{\triangleleft \mathbb{K}}^{p+q} \mathbf{x} \mathbb{L}_{\triangleleft \mathbb{K}}^p \mathbf{x} \mathbb{L}_{\triangleleft \mathbb{K}}^q$$

$${}_M\mathbb{L}\left(\mathbb{T}\mathbf{x}\mathbb{T}\right)^{\textcolor{teal}{l}}=\sum_{P\subset M}^{\sharp P=p}\overline{{}_P\mathbb{T}\mathbb{T}^{\textcolor{teal}{l}}}_{\overline{{}_P\mathbb{T}\mathbb{T}^{\textcolor{teal}{l}}}}\overline{{}_M\mathbb{T}\mathbb{T}^{\textcolor{teal}{l}}}_{\overline{{}_M\mathbb{T}\mathbb{T}^{\textcolor{teal}{l}}}}$$

$$\begin{bmatrix} \mathbb{L} \\ + \\ m\mathbb{L} \end{bmatrix} \mathbb{T} \mathbf{x} \mathbb{T}^{\textcolor{teal}{l}} = \sum_{1 \leqslant \nu_1 \leqslant \cdots \leqslant \nu_p \leqslant m} \overset{\nu}{-1} \begin{bmatrix} \nu_1 \mathbb{L} \\ + \\ \nu_p \mathbb{L} \end{bmatrix} \mathbb{T} \begin{bmatrix} \nu_{p+1} \mathbb{L} \\ + \\ \nu_m \mathbb{L} \end{bmatrix} \mathbb{T}^{\textcolor{teal}{l}}$$

$$\overset{\nu}{-1} = \overline{{}_P\mathbb{T}\mathbb{T}^{\textcolor{teal}{l}}}_{\overline{{}_P\mathbb{T}\mathbb{T}^{\textcolor{teal}{l}}}}$$

$$(\mathbf{x}\mathbb{L})\mathbf{x}(\mathbf{x}\mathbb{L}^{\textcolor{teal}{l}})=\mathbb{L}\mathbf{x}\mathbb{L}^{\textcolor{teal}{l}}$$

$$(\mathbf{x}\mathbb{L})\mathbf{x}\mathbb{T}=\mathbb{L}\mathbb{T}$$

$$\begin{aligned}
\mathbf{x} \perp &= \bigcap_{i \in I} (\mathbf{x} \perp_i) \\
(\mathbf{x} \perp) \bowtie (\mathbf{x} \perp') &= \perp \mathbf{x} \perp' \\
\top \bowtie \top' &= \left(\bigcap_{i \in I} \top \right) \bowtie \left(\bigcap_{j \in J} \top' \right) = \text{per} \left(\top \bowtie \top' \right) \\
\left(\bigcap_{i \in I} \perp \right) \mathbf{x} \left(\bigcap_{j \in J} \perp' \right) &= \text{per} \left(\perp \mathbf{x} \perp' \right) = \text{per} \left(\perp \left(\mathbf{x} \perp' \right) \right) = \left(\bigcap_{i \in I} \perp \right) \left(\bigcap_{j \in J} \left(\mathbf{x} \perp' \right) \right) \\
(\mathbf{x} (\mathbf{x} \perp_i)) \bowtie (\mathbf{x} \top^j) &= (\mathbf{x} (\mathbf{x} \perp_i)) \bowtie (\mathbf{x} \top^j) = (\mathbf{x} \perp_i) \left(\mathbf{x} \top^j \right) = \text{per} \left(\perp \top^j \right) = \text{per} \left((\mathbf{x} \perp_i) \bowtie \top^j \right) \\
&= \mathbb{K} \triangleleft \mathbb{L} \triangleleft \mathbb{K} \xleftarrow{?} \mathbb{L} \triangleleft \mathbb{K}
\end{aligned}$$

$$\begin{aligned}
\bigcap_{i \in I} \perp \bigcap_{j \in J} \top^j &= \text{per} \perp \top^j \\
\text{LHS} = m! \bigcap_{i \in I} \perp \bigcap_{j \in J} \top^j &= \sum_{I \xrightarrow{\sim} \pi J} -1 \bigcap_{i \in I} \perp \bigcap_{i \in I} \top^{\pi_i} = \sum_{\pi} \prod_{i \in I} \perp \top^{\pi_i} = \text{RHS}
\end{aligned}$$

$$\mathbb{L} \triangleleft \mathbb{K} \xrightleftharpoons[p_-]{\sqsubset} \mathbb{L} \triangleleft \mathbb{K}$$

$$p_- \top = \frac{1}{m!} \sum_{\pi \in \mathcal{C}|m} -1 \pi \bowtie \top$$

$$p_- (\tau \bowtie \top) = -1 p_- \top$$

$$p_- \left(p_- \top \right) = p_- \top$$

$$\left(p_- \top \right) \mathbf{x} \left(p_- \top' \right) = \begin{bmatrix} m \\ pq \end{bmatrix} p_- \left(\top \mathbf{x} \top' \right)$$

$$\begin{aligned}
m! \bigcap_{M} p_- \left(\top \mathbf{x} \top' \right) &= \sum_{\pi \in \mathcal{C}|M} -1 \bigcap_M \left(\top \mathbf{x} \top' \right) = \sum_{\pi \in \mathcal{C}|M} -1 \left(\bigcap_{\pi_{M<}} \top \right) \mathbf{x} \left(\bigcap_{\pi_{M>}} \top' \right) \\
&= \sum_{|P|=p} \overline{-1^{P>M\perp P}} \sum_{M_{<} \xrightarrow{\sim} \sigma P} -1 \left(\bigcap \top \right) \mathbf{x} \sum_{M_{>} \xrightarrow{\sim} \tau M\perp P} -1 \left(\bigcap \top' \right) = \sum_{|P|=p} \overline{-1^{P>M\perp P}} p! \left(\bigcap p_- \top \right) \mathbf{x} q! \left(\bigcap_{M\perp P} p_- \top' \right)
\end{aligned}$$

$$\begin{aligned}
\Leftarrow -1^{\pi} &= \prod_{M_{<} \ni i < j \in M_{<}} \pi_i \sharp \pi_j \prod_{M_{>} \ni i < j \in M_{>}} \pi_i \sharp \pi_j \prod_{M_{<} \ni i | j \in M_{>}} \pi_i \sharp \pi_j = -1^{\sigma} -1^{\tau} \overline{-1^{P>M\perp P}} | \sigma = \pi_{M_{<}} | \tau = \pi_{M_{>}} \\
p_- \left(\left(p_- \top \right) \mathbf{x} \top' \right) &= p_- \left(\top \mathbf{x} \top' \right) = p_- \top \mathbf{x} \left(p_- \top' \right)
\end{aligned}$$

$$\begin{bmatrix} m \\ pq \end{bmatrix} p_- \left(\left(p_- \lrcorner \right) \mathbf{x} \lrcorner \right) = p_- \left(\left(p_- \lrcorner \right) \mathbf{x} \left(p_- \lrcorner \right) \right) = \left(p_- \lrcorner \right) \mathbf{x} \left(p_- \lrcorner \right) = \begin{bmatrix} m \\ pq \end{bmatrix} p_- \lrcorner \mathbf{x} \left(p_- \lrcorner \right)$$

$$\mathbb{L}_{\triangleleft \triangleleft \mathbb{K}}^{\mathbb{N}} \text{ assoc super-comm}$$

$$\left(\lrcorner \mathbf{x} \lrcorner \right) \mathbf{x} \lrcorner = \begin{bmatrix} m \\ p+q:r \end{bmatrix} p_- \left(\left(\lrcorner \mathbf{x} \lrcorner \right) \mathbf{x} \lrcorner \right) = \begin{bmatrix} m \\ p:q:r \end{bmatrix} p_- p_- \left(\lrcorner \mathbf{x} \lrcorner \right) \mathbf{x} \lrcorner = \begin{bmatrix} m \\ p:q:r \end{bmatrix} p_- \left(\lrcorner \mathbf{x} \lrcorner \mathbf{x} \lrcorner \right)$$

$$\lrcorner \mathbf{x} \lrcorner = \begin{bmatrix} m \\ pq \end{bmatrix} p_- \left(\lrcorner \mathbf{x} \lrcorner \right) = \begin{bmatrix} m \\ pq \end{bmatrix} p_- \tau \ltimes \left(\lrcorner \mathbf{x} \lrcorner \right) = \lrcorner^{\tau} 1 \left(\lrcorner \mathbf{x} \lrcorner \right)$$

$$\Leftarrow \lrcorner^{\tau} 1 = \prod_{M_{<} \ni i < j \in M_{<}} \frac{\tau_i \sharp \tau_j}{8} (=1) \prod_{M_{>} \ni i < j \in M_{>}} \frac{\tau_i \sharp \tau_j}{8} (=1) \prod_{M_{<} \ni i:j \in M_{>}} \frac{\tau_i \sharp \tau_j}{8} (= -1) = \lrcorner^{pq} 1$$

$$\tau \begin{bmatrix} M_{<} \\ M_{>} \end{bmatrix} = \begin{bmatrix} M_{>} \\ M_{<} \end{bmatrix}$$

$$\lrcorner^0 \mathbf{x} \cdots \mathbf{x} \lrcorner^k = \begin{bmatrix} m \\ p_0 \cdots p_k \end{bmatrix} p_- \lrcorner^0 \mathbf{x} \cdots \mathbf{x} \lrcorner^k$$

$$\text{LHS} = \lrcorner^0 \mathbf{x} \left(\lrcorner^1 \mathbf{x} \cdots \mathbf{x} \lrcorner^k \right) = \begin{bmatrix} m \\ p_0:p_1 \cdots p_k \end{bmatrix} p_- \lrcorner^0 \mathbf{x} \left(\lrcorner^1 \mathbf{x} \cdots \mathbf{x} \lrcorner^k \right) = \begin{bmatrix} m \\ p_0:p_1 \cdots p_k \end{bmatrix} \begin{bmatrix} p_1 + \cdots + p_k \\ p_1 \cdots p_k \end{bmatrix} p_- \lrcorner^0 \mathbf{x} \left(p_- \lrcorner^1 \mathbf{x} \cdots \mathbf{x} \lrcorner^k \right) = \text{RHS}$$

$$\mathbb{L}_{\triangleleft \triangleleft \mathbb{K}}^m = \mathbb{L}_{\triangleleft \triangleleft \mathbb{K} \triangleleft \mathbb{K}}^m \ni \lrcorner^j = \bigcap_{j \in J} \mathbf{x} \lrcorner^j$$

$$\begin{array}{ccccc} \mathbb{L}_{\triangleleft \triangleleft \mathbb{K}}^m & \xrightleftharpoons[\mathbf{x}]{a} & \mathbb{K} \triangleleft \mathbb{N} \bar{\mathbb{L}} & \xrightleftharpoons[d]{c} & \bar{\mathbb{L}}_{\mathbb{N}} \triangleleft \mathbb{K} \\ \updownarrow \scriptstyle \begin{smallmatrix} m \\ * \\ f \end{smallmatrix} & & \updownarrow \scriptstyle \begin{smallmatrix} g \\ h \end{smallmatrix} & & \updownarrow \scriptstyle \begin{smallmatrix} i \\ j \end{smallmatrix} \\ \bar{\mathbb{L}}_{\triangleleft \triangleleft \mathbb{K}}^{n \bar{-} m} & \xrightleftharpoons[\mathbf{x}]{k} & \mathbb{K} \triangleleft \mathbb{N} \bar{\mathbb{L}} & \xrightleftharpoons[\lrcorner^N \mathbf{x}]{m} & \bar{\mathbb{L}}_{\mathbb{N}} \triangleleft \mathbb{K} \end{array}$$

$$* \left(\mathbf{x} \lrcorner^m \right) = \mathbb{L}^{n \bar{-} m} \lrcorner^N$$

$$\lrcorner^1 \mathbf{x} (* \lrcorner) = \overline{\lrcorner \mathbf{x} \lrcorner} \lrcorner^N$$

$$\lrcorner \mathbf{x} \lrcorner^1 = \mathbb{N} \lrcorner \overline{\lrcorner^1 \mathbf{x} (* \lrcorner)}$$

$$\lrcorner^1 \mathbf{x} \overline{(* \lrcorner)} = \lrcorner^1 \mathbf{x} \lrcorner^N = \lrcorner \lrcorner^1 \lrcorner^N = \overline{\lrcorner \lrcorner \lrcorner^1} \lrcorner^N$$

$$\begin{aligned}\overline{\mathfrak{X}\mathfrak{L}'\mathfrak{X}\ast\mathfrak{X}\mathfrak{L}} &= \overline{\mathfrak{X}\mathfrak{L}'\mathfrak{X}\mathfrak{L}\models\mathbb{1}^N} = \overline{\mathfrak{L}\mathfrak{X}\mathfrak{L}'\mathbb{1}^N} = \overline{\mathfrak{X}\mathfrak{L}\mathfrak{X}\mathfrak{L}'}\mathbb{1}^N \\ \mathfrak{L}\models\mathbb{1}^N &= \mathfrak{X}\overline{\mathbb{1}^N\mathfrak{X}\dashv\mathfrak{X}\mathfrak{L}} \\ \ast\mathbb{1} &= \mathfrak{X}\overline{\mathbb{1}^N\mathfrak{X}\dashv\mathbb{1}}\end{aligned}$$

$$\begin{aligned}\mathfrak{L}'\overline{\mathfrak{L}\models\mathbb{1}^N} &= \overline{\mathfrak{L}'\mathfrak{X}\mathfrak{L}}\mathbb{1}^N = \overline{\mathfrak{L}'\mathfrak{X}\mathfrak{L}}\mathfrak{X}\mathbb{1}^N\mathfrak{X} = \mathfrak{L}'\mathfrak{X}\overline{\mathbb{1}^N\mathfrak{X}\dashv\mathfrak{X}\mathfrak{L}} \\ \underbrace{{}_N\mathbb{1}\dashv\mathbb{1}^m}_{\mathbb{1}}\models\mathbb{1}^N &= \overset{m(n-m)}{-1}\mathbb{1}_N\overline{\mathbb{1}\mathbb{1}^N}\end{aligned}$$

$$\overset{m(n-m)}{-1}\mathfrak{L}\overline{{}_N\mathbb{1}\dashv\mathbb{1}\models\mathbb{1}^N}=\overset{m(n-m)}{-1}\overline{\mathfrak{L}\mathfrak{X}\mathbb{1}_N\mathbb{1}\dashv\mathbb{1}}\mathbb{1}^N=\overline{{}_N\mathbb{1}\dashv\mathbb{1}\mathfrak{X}\mathfrak{L}}\mathbb{1}^N=\overline{{}_N\mathbb{1}\dashv\mathbb{1}}\overline{\mathfrak{L}\models\mathbb{1}^N}=\overline{\mathbb{1}}\mathbb{1}_N\mathbb{1}\mathbb{1}^N$$

$$\overset{n-m}{\ast}\overset{m}{\ast}=\overset{m(n-m)}{-1}\mathbb{1}^N\mathfrak{X}\mathbb{1}^N$$

$$\overset{n-m}{\ast}\overset{m}{\ast}\overline{\mathfrak{X}\mathfrak{L}}=\overset{n-m}{\ast}\overline{\mathfrak{L}\models\mathbb{1}^N}=\overset{n-m}{\ast}\mathfrak{X}\overline{\mathbb{1}^N\mathfrak{X}\dashv\mathfrak{X}\mathfrak{L}}=\overline{\mathbb{1}^N\mathfrak{X}\dashv\mathfrak{X}\mathfrak{L}}\models\mathbb{1}^N=\overset{m(n-m)}{-1}\overline{\mathfrak{X}\mathfrak{L}}\mathbb{1}^N\mathfrak{X}\mathbb{1}^N$$

$$\mathbb{L}_{\triangleleft \mathbb{K}}^{\mathbb{N}} = \sum_m \mathbb{L}_{\triangleleft \mathbb{K}}^m \in \nabla_{\mathbb{K}}^{\mathbb{N}} \text{ Gel}$$

$$\mathbb{L}_{\triangleleft \mathbb{K}}^m = \frac{\mathbb{L}\mathfrak{X}\cdots\mathfrak{X}\mathbb{L}\xrightarrow{\mathbb{1}}\mathbb{K}}{\pi\mathfrak{X}\mathbb{1}=\overset{\pi}{-1}\mathbb{1}}\text{ anti-symm}$$

$$\mathbb{L}_{\triangleleft \mathbb{K}}^{p+q}\stackrel{\mathfrak{X}}{\longleftarrow}\mathbb{L}_{\triangleleft \mathbb{K}}^p\mathfrak{X}\mathbb{L}_{\triangleleft \mathbb{K}}^q$$

$${}_M\mathbb{L}\left(\overset{p}{\mathbb{1}}\mathfrak{X}\mathbb{1}'\right)=\sum_{P\subset M}^{\sharp P=p}\overline{{}^{\overline{P}>\overline{M\mathbb{L}P}}{-1}}\overline{{}_P\mathbb{L}}\overline{{}_{M\mathbb{L}P}\mathbb{L}'}'$$

$$\begin{aligned}\begin{bmatrix}\mathbb{1} \\ + \\ m\mathbb{1}\end{bmatrix}\mathbb{1}\mathfrak{X}\mathbb{1}' &= \sum_{1\leqslant \nu_1<\cdots<\nu_p\leqslant m}\overset{\nu}{-1}\begin{bmatrix}\nu_1\mathbb{1} \\ + \\ \nu_p\mathbb{1}\end{bmatrix}\mathbb{1}\begin{bmatrix}\nu_{p+1}\mathbb{1} \\ + \\ \nu_m\mathbb{1}\end{bmatrix}\mathbb{1}' \\ -1 &= \overline{{}^{\overline{P}>\overline{M\mathbb{L}P}}{-1}}\end{aligned}$$

$$(\mathfrak{X}\mathfrak{L})\mathfrak{X}\Big(\mathfrak{X}\mathfrak{L}'\Big):=\mathfrak{L}\mathfrak{X}\mathfrak{L}'$$

$$(\mathfrak{X}\mathfrak{L})\mathfrak{X}\mathbb{1}=\mathfrak{L}\mathbb{1}$$

$$\mathfrak{X}\mathbb{1}=\mathop{i\in I}\nolimits\mathfrak{X}(\mathfrak{X}\mathbb{1}_i)$$

$$(\mathfrak{X}\mathbb{1}_i)\mathfrak{X}\Big(\mathfrak{X}\mathbb{1}_j'\Big)=\mathbb{1}_i\mathfrak{X}\mathbb{1}_j'$$

$$\mathbb{1}'\mathfrak{X}\mathbb{1}'=\Big(\mathop{i\in I}\nolimits\mathfrak{X}\mathbb{1}'^i\Big)\mathfrak{X}\Big(\mathop{j\in J}\nolimits\mathfrak{X}\mathbb{1}'^j\Big)=\det\Big(\mathbb{1}'\mathfrak{X}\mathbb{1}'^j\Big)$$

$$\begin{aligned}
& \left(\underset{i \in I}{\mathbb{X}} \underset{i}{\mathbb{L}} \right) \ltimes \left(\underset{j \in J}{\mathbb{X}} \underset{j}{\mathbb{L}} \right) = \det \left(\underset{i}{\mathbb{L}} \ltimes \underset{j}{\mathbb{L}} \right) = \det \left(\underset{i}{\mathbb{L}} \left(\ltimes \underset{j}{\mathbb{L}} \right) \right) = \left(\underset{i \in I}{\mathbb{X}} \underset{i}{\mathbb{L}} \right) \left(\underset{j \in J}{\mathbb{X}} \left(\ltimes \underset{j}{\mathbb{L}} \right) \right) \\
& \left(\ltimes \left(\ltimes \underset{i}{\mathbb{L}} \right) \right) \ltimes \left(\ltimes \underset{j}{\mathbb{L}} \right) = \left(\ltimes \left(\ltimes \underset{i}{\mathbb{L}} \right) \right) \ltimes \left(\ltimes \underset{j}{\mathbb{L}} \right) = \left(\ltimes \underset{i}{\mathbb{L}} \right) \left(\ltimes \underset{j}{\mathbb{L}} \right) = \det \left(\underset{i}{\mathbb{L}} \underset{j}{\mathbb{L}} \right) = \det \left(\left(\ltimes \underset{i}{\mathbb{L}} \right) \ltimes \underset{j}{\mathbb{L}} \right) \\
& = \mathbb{K} \ltimes \mathbb{L} \xrightarrow{p_-} \mathbb{K} \xleftarrow{?} \mathbb{L} \xrightarrow{p_-} \mathbb{K} \\
& \underset{i \in I}{\mathbb{X}} \underset{i}{\mathbb{L}} \mid \underset{j \in J}{\mathbb{X}} \underset{j}{\mathbb{L}} = \det \underset{i}{\mathbb{L}} \underset{j}{\mathbb{L}}
\end{aligned}$$

$$\text{LHS} = m! \underset{i \in I}{\mathbb{X}} \underset{i}{\mathbb{L}} \mid p_- \underset{j \in J}{\mathbb{X}} \underset{j}{\mathbb{L}} = \sum_{I \xrightarrow{\sim} \pi J} -1 \underset{i \in I}{\mathbb{X}} \underset{i}{\mathbb{L}} \mid \underset{i \in I}{\mathbb{X}} \underset{i}{\mathbb{L}} = \sum_{\pi} \prod_{i \in I} \underset{i}{\mathbb{L}} = \text{RHS}$$

$$\mathbb{L} \xrightarrow[p_-]{\mathbb{K}} \mathbb{L} \xrightarrow[p_-]{\mathbb{K}}$$

$$p_- \mathbb{L} = \frac{1}{m!} \sum_{\pi \in \mathbb{C}|m} -1 \pi \ltimes \mathbb{L}$$

$$p_- (\tau \ltimes \mathbb{L}) = -1 p_- \mathbb{L}$$

$$p_- (p_- \mathbb{L}) = p_- \mathbb{L}$$

$$(p_- \mathbb{L}) \ltimes (p_- \mathbb{L}) = \begin{bmatrix} m \\ pq \end{bmatrix} p_- (\mathbb{L} \ltimes \mathbb{L})$$

$$m! \underset{M}{\mathbb{L}} p_- (\mathbb{L} \ltimes \mathbb{L}) = \sum_{\pi \in \mathbb{C}|M} -1 \underset{\pi_M}{\mathbb{L}} (\mathbb{L} \ltimes \mathbb{L}) = \sum_{\pi \in \mathbb{C}|M} -1 \left(\underset{\pi_{M<}}{\mathbb{L}} \right) \ltimes \left(\underset{\pi_{M>}}{\mathbb{L}} \right)$$

$$= \sum_{|P|=p} \overline{-1}^{P>M\ltimes P} \sum_{M_{<} \xrightarrow{\sim} \sigma P} -1 (\mathbb{L} \ltimes \mathbb{L}) \ltimes \sum_{M_{>} \xrightarrow{\sim} \tau M\ltimes P} -1 (\mathbb{L} \ltimes \mathbb{L}) = \sum_{|P|=p} \overline{-1}^{P>M\ltimes P} p! (\mathbb{L} p_- \mathbb{L}) \ltimes q! (\underset{M\ltimes P}{\mathbb{L}} p_- \mathbb{L})$$

$$\Leftarrow -1 = \prod_{M_{<} \ni i < j \in M_{<}} \pi_i \sharp \pi_j \prod_{M_{>} \ni i < j \in M_{>}} \pi_i \sharp \pi_j \prod_{M_{<} \ni i:j \in M_{>}} \pi_i \sharp \pi_j = -1 -1 \overline{-1}^{P>M\ltimes P} | \sigma = \pi_{M_{<}} | \tau = \pi_{M_{>}}$$

$$p_- ((p_- \mathbb{L}) \ltimes \mathbb{L}) = p_- (\mathbb{L} \ltimes \mathbb{L}) = p_- \mathbb{L} \ltimes (p_- \mathbb{L})$$

$$\begin{bmatrix} m \\ pq \end{bmatrix} p_- ((p_- \mathbb{L}) \ltimes \mathbb{L}) = p_- ((p_- \mathbb{L}) \ltimes (p_- \mathbb{L})) = (p_- \mathbb{L}) \ltimes (p_- \mathbb{L}) = \begin{bmatrix} m \\ pq \end{bmatrix} p_- \mathbb{L} \ltimes (p_- \mathbb{L})$$

$$\mathbb{L} \xrightarrow[p_-]{\mathbb{K}} \mathbb{K} \text{ assoc super-comm}$$

$$(\mathbb{L} \ltimes \mathbb{L}) \ltimes \mathbb{L} = \begin{bmatrix} m \\ p+q:r \end{bmatrix} p_- ((\mathbb{L} \ltimes \mathbb{L}) \ltimes \mathbb{L}) = \begin{bmatrix} m \\ p:q:r \end{bmatrix} p_- p_- (\mathbb{L} \ltimes \mathbb{L}) \ltimes \mathbb{L} = \begin{bmatrix} m \\ p:q:r \end{bmatrix} p_- (\mathbb{L} \ltimes \mathbb{L} \ltimes \mathbb{L})$$

$$\mathbb{T}\mathbf{x}\acute{\mathbb{T}}=\begin{bmatrix}m\\pq\end{bmatrix}p_{-}\left(\mathbb{T}\mathbf{x}\acute{\mathbb{T}}\right)=\begin{bmatrix}m\\pq\end{bmatrix}p_{-}\tau\ltimes\left(\acute{\mathbb{T}}\mathbf{x}\mathbb{T}\right)=-^{\tau}1\left(\acute{\mathbb{T}}\mathbf{x}\mathbb{T}\right)$$

$$\Leftarrow -^{\tau}1=\prod_{M_{<}\ni i<j\in M_{<}}\frac{\tau_i\sharp\tau_j}{8}(=1)\prod_{M_{>}\ni i<j\in M_{>}}\frac{\tau_i\sharp\tau_j}{8}(=1)\prod_{M_{<}\ni i:j\in M_{>}}\frac{\tau_i\sharp\tau_j}{8}(=-1)=-^{pq}1$$

$$\tau\begin{bmatrix}M_{<}\\M_{>}\end{bmatrix}=\begin{bmatrix}M_{>}\\M_{<}\end{bmatrix}$$

$$\mathbb{T}^0\mathbf{x}\cdots\mathbf{x}\mathbb{T}^k=\begin{bmatrix}m\\p_0\cdots p_k\end{bmatrix}p_{-}\mathbb{T}^0\mathbf{x}\cdots\mathbf{x}\mathbb{T}^k$$

$$\text{LHS} \, = \, \mathbb{T}^0\mathbf{x}\Big(\mathbb{T}^1\mathbf{x}\cdots\mathbf{x}\mathbb{T}^k\Big) \, = \, \begin{bmatrix}m\\p_0;p_1\cdots p_k\end{bmatrix}p_{-}\mathbb{T}^0\mathbf{x}\Big(\mathbb{T}^1\mathbf{x}\cdots\mathbf{x}\mathbb{T}^k\Big) \, = \, \begin{bmatrix}m\\p_0;p_1\cdots p_k\end{bmatrix}\begin{bmatrix}p_1+\cdots+p_k\\p_1\cdots p_k\end{bmatrix}p_{-}\mathbb{T}^0\mathbf{x}\Big(p_{-}\mathbb{T}^1\mathbf{x}\cdots\mathbf{x}\mathbb{T}^k\Big) \, = \, \text{RHS}$$

$$\mathbb{L}_{\blacktriangleleft\overline{\mathbb{K}}}^{\overline{m}}=\mathbb{L}_{\blacktriangleleft\mathbb{K}\blacktriangleleft\overline{\mathbb{K}}}^{\overline{m}}\ni\mathbb{T}^j=\underset{j\in J}{\mathbf{x}}\mathbb{T}^i$$

$$\begin{array}{ccccc} \mathbb{L}_{\blacktriangleleft\overline{\mathbb{K}}}^{\overline{m}} & \overset{a}{\underset{\mathbf{x}}{\rightleftarrows}} & \mathbb{K}_{\overline{m}}\blacktriangledown\overline{\mathbb{L}} & \overset{c}{\underset{d}{\rightleftarrows}} & \vartheta|\overline{\mathbb{L}}_{\blacktriangleleft\overline{\mathbb{K}}}^{\overline{N}} \\ \updownarrow \overset{\overline{m}}{\ast} \updownarrow f & & \updownarrow \overset{g}{\downarrow} \updownarrow h & & \updownarrow \overset{i}{\downarrow} \updownarrow j \\ \overline{\mathbb{L}}_{\blacktriangleleft\overline{\mathbb{K}}}^{\overline{n-m}} & \overset{k}{\underset{\mathbf{x}}{\rightleftarrows}} & \mathbb{K}_{\overline{n-m}}\blacktriangledown\overline{\mathbb{L}} & \overset{m}{\underset{\mathbb{T}^N\mathbf{x}}{\rightleftarrows}} & \vartheta|\overline{\mathbb{K}}_{\overline{N}}\blacktriangledown\overline{\mathbb{L}} \end{array}$$

$$\ast\left(\mathbf{x}\overline{\mathbb{L}}\right)=\mathbb{L}^{\overline{n-m}}\models\mathbb{T}^N$$

$$\acute{\mathbb{T}}\mathbf{x}(\ast\mathbb{T})=\overline{\mathbb{T}\mathbf{x}\acute{\mathbb{T}}}\mathbb{T}^N$$

$$\mathbb{T}\mathbf{x}\acute{\mathbb{T}}=\overline{{}_N\mathbb{T}\acute{\mathbb{T}}\mathbf{x}(\ast\mathbb{T})}$$

$$\acute{\mathbb{T}}\mathbf{x}\overline{\ast\mathbf{x}\overline{\mathbb{L}}}=\acute{\mathbb{T}}\mathbf{x}\overline{\mathbb{L}\models\mathbb{T}^N}=\overline{\mathbb{L}\acute{\mathbb{T}}}\mathbb{T}^N=\overline{\mathbf{x}\overline{\mathbb{L}\mathbf{x}\acute{\mathbb{T}}}}\mathbb{T}^N$$

$$\overline{\mathbf{x}\overline{\mathbb{L}}}\mathbf{x}\overline{\ast\mathbf{x}\overline{\mathbb{L}}}=\overline{\mathbf{x}\overline{\mathbb{L}}}\mathbf{x}\overline{\mathbb{L}\models\mathbb{T}^N}=\overline{\mathbb{L}\mathbf{x}\overline{\mathbb{L}}}\mathbb{T}^N=\overline{\mathbf{x}\overline{\mathbb{L}\mathbf{x}\overline{\mathbf{x}\overline{\mathbb{L}}}}}\mathbb{T}^N$$

$$\mathbb{L}\models\mathbb{T}^N=\mathbf{x}\overline{\overline{\mathbb{T}^N\mathbf{x}\dashv}\dashv\mathbf{x}\overline{\mathbb{L}}}$$

$$\ast\mathbb{T}=\mathbf{x}\overline{\overline{\mathbb{T}^N\mathbf{x}\dashv}\dashv\mathbb{T}}$$

$$\overline{\mathbb{L} \sqcup \models \mathbb{T}^N} = \overline{\mathbb{L} \boxtimes \mathbb{L}} \mathbb{T}^N = \overline{\mathbb{L} \boxtimes \mathbb{L}} \star \overline{\mathbb{T}^N \boxtimes} = \overline{\mathbb{L} \star \overline{\mathbb{T}^N \boxtimes} \dashv \star \mathbb{L}}$$

$$\overline{\mathbb{T} \dashv \mathbb{T}} \models \mathbb{T}^N = \overset{m(n-m)}{-1} \mathbb{T} \overline{\mathbb{T} \mathbb{T}^N}$$

$$\overset{m(n-m)}{-1} \overline{\mathbb{L} \overline{\mathbb{T} \dashv \mathbb{T}} \models \mathbb{T}^N} = \overset{m(n-m)}{-1} \overline{\mathbb{L} \boxtimes \overline{\mathbb{T} \dashv \mathbb{T}}} \mathbb{T}^N = \overline{\mathbb{T} \dashv \mathbb{T} \boxtimes \mathbb{L}} \mathbb{T}^N = \overline{\mathbb{T} \dashv \mathbb{T}} \overline{\mathbb{L} \models \mathbb{T}^N} = \overline{\mathbb{L}} \mathbb{T} \mathbb{T}^N$$

$$\overset{n-m}{\star} \overset{m}{\star} = \overset{m(n-m)}{-1} \mathbb{T}^N \star \mathbb{T}^N$$

$$\overset{n-m}{\star} \overset{m}{\star} \overline{\star \mathbb{L}} = \overset{n-m}{\star} \overline{\mathbb{L} \models \mathbb{T}^N} = \overset{n-m}{\star} \star \overline{\mathbb{T}^N \boxtimes \dashv \star \mathbb{L}} = \overline{\mathbb{T}^N \boxtimes \dashv \star \mathbb{L}} \models \mathbb{T}^N = \overset{m(n-m)}{-1} \overline{\star \mathbb{L}} \mathbb{T}^N \star \mathbb{T}^N$$