$$\sum_{n\geqslant 1}\frac{1}{n}=+\infty$$

$$1+1/2+1/3+1/4+1/5+1/6+1/7+1/8+1/9+$$

$$=1+1/2+\underbrace{1/3+1/4}_{\geqslant 2\cdot 1/4}+\underbrace{1/5+1/6+1/7+1/8}_{\geqslant 4\cdot 1/8}+\underbrace{1/9++1/16}_{\geqslant 8\cdot 1/16}+\geqslant 1+1/2+1/2+1/2+=+\infty$$

$$s>1\Rightarrow\frac{1-2^{-s}}{1-2^{1-s}}\leqslant\zeta(s)=\sum_{n\geqslant 1}\frac{1}{n^s}\leqslant\frac{1}{1-2^{1-s}}<+\infty$$

$$\text{n.o. :}\quad 1+(1/2)^s+(1/3)^s+(1/4)^s+(1/5)^s+(1/6)^s+(1/7)^s+(1/8)^s+(1/9)^s+$$

$$=1+\underbrace{(1/2)^s+(1/3)^s}_{\leqslant 2\cdot(1/2)^s}+\underbrace{(1/4)^s+(1/5)^s+(1/6)^s+(1/7)^s}_{\leqslant 4\cdot(1/4)^s}+\underbrace{(1/8)^s++(1/15)^s}_{\leqslant 8\cdot(1/8)^s}+$$

$$\leqslant 1+(1/2)^{s-1}+(1/4)^{s-1}+(1/8)^{s-1}+=1+2^{1-s}+4^{1-s}+8^{1-s}+$$

$$=\sum_{n}^{\mathbb{N}}\left(2^n\right)^{1-s}=\sum_{n}^{\mathbb{N}}2^{n(1-s)}=\sum_{n}^{\mathbb{N}}\left(2^{1-s}\right)^n=\frac{1}{1-2^{1-s}}$$

$$\text{n.u. :}\quad 1+(1/2)^s+(1/3)^s+(1/4)^s+(1/5)^s+(1/6)^s+(1/7)^s+(1/8)^s+(1/9)^s+$$

$$=1+(1/2)^s+\underbrace{(1/3)^s+(1/4)^s}_{\geqslant 2\cdot(1/4)^s}+\underbrace{(1/5)^s+(1/6)^s+(1/7)^s+(1/8)^s}_{\geqslant 4\cdot(1/8)^s}+\underbrace{(1/9)^s++(1/16)^s}_{\geqslant 8\cdot(1/16)^s}+$$

$$\geqslant 1+(1/2)^s+2(1/4)^s+4(1/8)^s+=1+2^{-s}+4^{-s}2+8^{-s}4+=\frac{1}{2}+\frac{1+2^{1-s}+4^{1-s}+8^{1-s}+}{2}$$

$$=\frac{1}{2}+\frac{1}{2}\sum_{n}^{\mathbb{N}}\left(2^n\right)^{1-s}=\frac{1}{2}+\frac{1}{2}\sum_{n}^{\mathbb{N}}2^{n(1-s)}=\frac{1}{2}+\frac{1}{2}\sum_{n}^{\mathbb{N}}\left(2^{1-s}\right)^n=\frac{1}{2}+\frac{1/2}{1-2^{1-s}}=\frac{1-2^{-s}}{1-2^{1-s}}$$

$$0<n^{-s}\searrow 0:\quad\sum_{0\leqslant m}2^m\left(2^m\right)^{-s}=\sum_{0\leqslant m}\left(2^{1-s}\right)^m=\frac{1}{1-2^{1-s}}=\frac{2^{s-1}}{2^{s-1}-1}<\infty$$

$$\frac{3}{2}=\frac{1-2^{-2}}{1-2^{-1}}\leqslant\zeta(2)=\sum_{n\geqslant 1}\frac{1}{n^2}\leqslant\frac{1}{1-2^{-1}}=2$$

$$\zeta\left(2\right) = \sum_{n \geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\frac{\zeta\left(2\right)}{\left(2\pi\right)^2} = \sum_{n \geqslant 1} \frac{1}{\left(2\pi n\right)^2} = \frac{1}{24} \in \mathbb{Q}$$

$$\frac{\zeta\left(2k\right)}{\left(2\pi\right)^{2k}} \underset{\text{Euler}}{=} \sum_{n \geqslant 1} \frac{1}{\left(2\pi n\right)^{2k}} = \frac{1}{2}\frac{\left|B_{2k}\right|}{\left(2k\right)!} \in \mathbb{Q}$$

$$a_n \searrow 0 \Rightarrow \sum_{1 \leqslant n} a_n \leqslant \sum_{0 \leqslant m} a_{2^m} 2^m \leqslant 2 \sum_{1 \leqslant n} a_n$$

$$b_m = a_{2^m} 2^m \text{ verdichtete Folge}$$

$$\bar{a}_N = \sum_{1 \leqslant n \leqslant N} a_n$$

$$\bar{a}_{2^M - 1} \leqslant \bar{b}_{M-1}$$

$$\text{LHS} = a_1 + + a_{2^M - 1} \overset{\text{links}}{\underset{\text{klam}}{=}} \underbrace{a_1}_{2^0} + \underbrace{a_2 + a_3}_{2^1} + \underbrace{a_4 + a_5 + a_6 + a_7}_{2^2} + + \underbrace{a_{2^{M-1}} + a_{2^{M-1}+1} + + a_{2^M - 1}}_{2^{M-1}}$$

$$= \sum_m^M \sum_n^{2^m} \underbrace{a_{n + 2^m}}_{\leqslant a_{2^m}} \leqslant \sum_m^M \sum_n^{2^m} a_{2^m} = \sum_m^M a_{2^m} 2^m = \sum_m^M b_m = \text{RHS}$$

$$\bar{a}_\infty = \sum_{1 \leqslant n} a_n = \curlyvee_N \bar{a}_{N-1} = \curlyvee_M \bar{a}_{2^M - 1} \leqslant \curlyvee_M \bar{b}_{M-1} = \sum_{0 \leqslant m} b_m = \bar{b}_\infty$$

$$\bar{a}_{2^M} \geqslant \bar{b}_M / 2$$

$$\text{LHS} = a_1 + + a_{2^M} \overset{\text{rechts}}{\underset{\text{klammern}}{=}} a_1 + \underbrace{a_2}_{2^0} + \underbrace{a_3 + a_4}_{2^1} + \underbrace{a_5 + a_6 + a_7 + a_8}_{2^2} + + \underbrace{a_{1 + 2^{M-1}} + + a_{2^M}}_{2^{M-1}}$$

$$= \underbrace{a_1}_{\geqslant a_1/2} + \sum_m^M \sum_n^{2^m} \underbrace{a_{n + 1 + 2^m}}_{\geqslant a_{2^{1+m}}} \geqslant \frac{a_1}{2} + \sum_m^M \sum_n^{2^m} a_{2^{1+m}} = \frac{a_1}{2} + \sum_m^M a_{2^{1+m}} 2^m$$

$$= \frac{1}{2} \left(a_1 + \sum_m^M a_{2^{1+m}} 2^{1+m} \right) \underset{k = 1 + m}{=} \frac{1}{2} \sum_k^{1+M} a_{2^k} 2^k = \frac{1}{2} \sum_k^{1+M} b_k = \text{RHS}$$

$$\bar{b}_\infty = \sum_{0 \leqslant m} b_m = \curlyvee_M \bar{b}_M \leqslant 2 \curlyvee_M \bar{a}_{2^M} = 2 \curlyvee_N \bar{a}_N = 2 \sum_{1 \leqslant n} a_n = \bar{a}_\infty$$