$$\text{difference relation } R \text{ on } \mathbb{1}\times\mathbb{4} = \frac{m{:}p}{m \in \mathbb{1}: \quad p \in \mathbb{4}}$$

$$m{:}p \sim n{:}q \Leftrightarrow \bigvee_{b \in \mathbb{4}} \underbrace{m+q} + b = \underbrace{n+p} + b \text{ equ rel}$$

$$\text{refl } m+p = m+p \Longrightarrow m{:}p \sim m{:}p$$

$$\text{symm } m{:}p \sim n{:}q \Longrightarrow m+q+b = n+p+b \Longrightarrow n+p+b = m+q+b \Longrightarrow n{:}q \sim m{:}p$$

$$\text{trans } m{:}p \sim n{:}q \sim r{:}s \Longrightarrow \begin{cases} \bigvee\limits_{b \in \mathbb{4}} & m+q+b = n+p+b \\ \bigvee\limits_{d \in \mathbb{4}} & n+s+d = r+q+d \end{cases}$$

$$\Longrightarrow \underbrace{m+s} + \underbrace{q+b+d} = \underbrace{m+q+b} + \underbrace{s+d} \underset{\text{V0r}}{=} \underbrace{n+p+b} + \underbrace{s+d}$$

$$= \underbrace{n+s+d} + \underbrace{p+b} \underset{\text{V0r}}{=} \underbrace{r+q+d} + \underbrace{p+b} = \underbrace{r+p} + \underbrace{q+b+d}$$

$$\underset{q+b+d \in \mathbb{4}}{\Longrightarrow} m{:}p \sim r{:}s$$

$$\text{difference class } m \ominus p$$

$$m{:}p \sim n{:}q \Leftrightarrow m \ominus p \sim n \ominus q$$

$$\mathbb{1} \ominus \mathbb{4} = \underbrace{\mathbb{1}\times\mathbb{4}} \ulcorner R = \frac{m \ominus p}{m \in \mathbb{1}: \quad p \in \mathbb{4}} \text{ difference ring}$$

$$m \ominus p \in \mathbb{1} \ominus \mathbb{4} \xleftarrow[\text{surj}]{\ominus} \mathbb{1}\times\mathbb{4} \ni m{:}p$$

$$\text{add differences } \underbrace{\mathbb{1} \ominus \mathbb{4}} \times \underbrace{\mathbb{1} \ominus \mathbb{4}} \xrightarrow{+} \mathbb{1} \ominus \mathbb{4}$$

$$\underbrace{m \ominus p} + \underbrace{n \ominus q} = \underbrace{m+n} \ominus \underbrace{p+q} \text{ well-def}$$

$$\begin{cases} m \ominus p = \acute{m} \ominus \acute{p} & \Longrightarrow m + \acute{p} + b = \acute{m} + p + b \\ n \ominus q = \acute{n} \ominus \acute{q} & \Longrightarrow n + \acute{q} + d = \acute{n} + q + d \end{cases} \Longrightarrow$$

$$\overbrace{\underbrace{m+n} + \underbrace{\acute{p}+\acute{q}}} + \underbrace{b+d} = \underbrace{m+\acute{p}+b} + \underbrace{n+\acute{q}+d} \underset{\text{V0r}}{=} \underbrace{\acute{m}+p+b} + \underbrace{\acute{n}+q+d} = \overbrace{\underbrace{\acute{m}+\acute{n}} + \underbrace{p+q}} + \underbrace{b+d}$$

$$\overset{\text{def}}{\underset{b+d \in \mathbb{4}}{\Longrightarrow}} \underbrace{m+n}{:}\underbrace{p+q} \sim \underbrace{\acute{m}+\acute{n}}{:}\underbrace{\acute{p}+\acute{q}} \Longrightarrow \underbrace{m+n} \ominus \underbrace{p+q} = \underbrace{\acute{m}+\acute{n}} \ominus \underbrace{\acute{p}+\acute{q}}$$

multiply differences $\underbrace{\mathbb{1}\ominus\mathbb{4}}\times\underbrace{\mathbb{1}\ominus\mathbb{4}} \xrightarrow{\times} \mathbb{1}\ominus\mathbb{4}$

$\underbrace{m\ominus p}\times\underbrace{n\ominus q} = \underbrace{m\times n+p\times q}\ominus\underbrace{m\times q+p\times n}$ well-def

$$\begin{cases} m\ominus p = \acute{m}\ominus\acute{p} & \Longrightarrow \underbrace{m+\acute{p}}+b = \underbrace{\acute{m}+p}+b \\ n\ominus q = \acute{n}\ominus\acute{q} & \Longrightarrow \underbrace{n+\acute{q}}+d = \underbrace{\acute{n}+q}+d \end{cases}$$

$$\Longrightarrow \overbrace{\underbrace{m\times n+p\times q} + \underbrace{\acute{m}\times\acute{q}+\acute{p}\times\acute{n}}} + \overbrace{\underbrace{b+\acute{p}}\times n + \underbrace{\acute{m}+\acute{p}}\times d + b\times q}$$

$$= \underbrace{m+\acute{p}+b}\times n + \underbrace{b+p}\times q + \acute{p}\times\underbrace{\acute{n}+d} + \acute{m}\times\underbrace{d+\acute{q}} \underset{\text{V0r}}{=} \underbrace{\acute{m}+p+b}\times n + \underbrace{b+p}\times q + \acute{p}\times\underbrace{\acute{n}+d} + \acute{m}\times\underbrace{d+\acute{q}}$$

$$= \acute{m}\times\underbrace{n+\acute{q}+d} + \acute{p}\times\underbrace{\acute{n}+d} + \underbrace{b+p}\times\underbrace{q+n} \underset{\text{V0r}}{=} \acute{m}\times\underbrace{\acute{n}+q+d} + \acute{p}\times\underbrace{\acute{n}+d} + \underbrace{b+p}\times\underbrace{q+n}$$

$$= \underbrace{\acute{m}+p+b}\times q + \underbrace{p+b}\times n + \underbrace{\acute{m}+\acute{p}}\times\underbrace{d+\acute{n}} \underset{\text{V0r}}{=} \underbrace{m+\acute{p}+b}\times q + \underbrace{p+b}\times n + \underbrace{\acute{m}+\acute{p}}\times\underbrace{d+\acute{n}}$$

$$= \acute{p}\times\underbrace{\acute{n}+q+d} + \underbrace{m+b}\times q + \acute{m}\times\underbrace{\acute{n}+d} + \underbrace{p+b}\times n \underset{\text{V0r}}{=} \acute{p}\times\underbrace{n+\acute{q}+d} + \underbrace{m+b}\times q + \acute{m}\times\underbrace{\acute{n}+d} + \underbrace{p+b}\times n$$

$$= \overbrace{\underbrace{\acute{m}\times\acute{n}+\acute{p}\times\acute{q}} + \underbrace{m\times q+p\times n}} + \overbrace{\underbrace{b+\acute{p}}\times n + \underbrace{\acute{m}+\acute{p}}\times d + b\times q}$$

$$\underset{\underbrace{b+\acute{p}}\times n+\underbrace{\acute{m}+\acute{p}}\times d+b\times q\,\in\,\mathbb{4}}{\overset{\text{def}}{\Longrightarrow}} \underbrace{m\times n+p\times q}:\underbrace{m\times q+p\times n} \sim \underbrace{\acute{m}\times\acute{n}+\acute{p}\times\acute{q}}:\underbrace{\acute{m}\times\acute{q}+\acute{p}\times\acute{n}}$$

$$\Longrightarrow \underbrace{m\times n+p\times q}\ominus\underbrace{m\times q+p\times n} = \underbrace{\acute{m}\times\acute{n}+\acute{p}\times\acute{q}}\ominus\underbrace{\acute{m}\times\acute{q}+\acute{p}\times\acute{n}}$$

$$\text{trivial difference}\quad \begin{tikzcd} m\ominus 0 \in \mathbb{1}\ominus\mathbb{4} & \mathbb{1}\times\mathbb{4} \arrow[l, "\ominus"'] & \mathbb{1}\ni m \arrow[l, ":0"'] \arrow[ll, bend left, "\ominus 0"] \end{tikzcd}$$

$\mathbb{4}$ kuerzbar $\Longrightarrow$ inj

$$/ \qquad \underbrace{m\ominus 0}+\underbrace{n\ominus 0} = \underbrace{m+n}\ominus 0$$

$$\underbrace{m\ominus 0}+\underbrace{n\ominus 0} = \underbrace{m+n}\ominus\underbrace{0+0} = \underbrace{m+n}\ominus 0$$

$$// \qquad \underbrace{m\ominus 0}\times\underbrace{n\ominus 0} = \underbrace{m\times n}\ominus 0$$

$$\underbrace{m\ominus 0}\times\underbrace{n\ominus 0} = \underbrace{m\times n+0\times 0}\ominus\underbrace{m\times 0+0\times n} = \underbrace{m\times n+0}\ominus\underbrace{0+0} = \underbrace{m\times n}\ominus 0$$