$$\frac{1}{\tau} + \sum_{m}^{\mathbb{Z}^\times} \left(\frac{1}{\tau + m} - \frac{1}{m} \right) = \pi \cot \pi\tau = \pi i \left(1 + 2 \sum_{d \geqslant 1} \mathfrak{e}^{2\pi i \tau d} \right)$$

$$q = \mathfrak{e}^{2\pi i \tau}$$

$$-i \cot \pi\tau = -i \frac{{}^{\pi\tau}\mathfrak{c}}{{}^{\pi\tau}\mathfrak{s}} = \frac{{}^{-\pi i \tau}\mathfrak{e} + {}^{\pi i \tau}\mathfrak{e}}{{}^{-\pi i \tau}\mathfrak{e} - {}^{\pi i \tau}\mathfrak{e}} = \frac{1 + {}^{2\pi i \tau}\mathfrak{e}}{1 - {}^{2\pi i \tau}\mathfrak{e}}$$

$$= \frac{1+q}{1-q} = \frac{1}{1-q} + \frac{q}{1-q} = \sum_{d}^{\mathbb{N}} q^d + \sum_{d}^{\mathbb{N}} q^{d+1} = 1 + 2 \sum_{d \geqslant 1} q^d$$

$$k \geqslant 1 \Longrightarrow \overset{k}{-}1 \sum_{m}^{\mathbb{Z}} \frac{k!}{(\tau + m)^{k+1}} = (2\pi i)^{k+1} \sum_{d \geqslant 1} d^k \, \mathfrak{e}^{2\pi i \tau d}$$

$$(\ell)^k = \sum_{d \prec \ell} d^k$$

$$1 \leqslant k \text{ ev } \Longrightarrow \frac{1}{2} \underbrace{\mathbb{Z} \overset{\times}{+} \tau\mathbb{Z}}_{-k} = {}^{k}\zeta - \frac{(2\pi i)^k}{(k-1)!} \sum_{\ell \geqslant 1} (\ell)^{k-1} \, \mathfrak{e}^{2\pi i \ell \tau}$$

$$\text{LHS} = \frac{1}{2} \sum_{n}^{\mathbb{Z}^\times} \left(\overset{-k}{n} + \sum_{m}^{\mathbb{Z}} \overset{-k}{\overbrace{m + \tau n}} \right) \underset{\text{k ev}}{=} \sum_{n \geqslant 1} \left(\overset{-k}{n} + \sum_{m}^{\mathbb{Z}} \overset{-k}{\overbrace{m + \tau n}} \right)$$

$$= {}^{k}\zeta - \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geqslant 1} \sum_{d \geqslant 1} d^{k-1} \, \mathfrak{e}^{2\pi i n \tau d} \underset{\ell = nd}{=} {}^{k}\zeta - \frac{(2\pi i)^k}{(k-1)!} \sum_{\ell \geqslant 1} \overbrace{\sum_{nd = \ell} d^{k-1}}^{= (\ell)^{k-1}} \mathfrak{e}^{2\pi i \ell \tau} = \text{RHS}$$

$$12 \prec 5d^2 + 7d^4$$

$$5d^2 + 7d^4 = d^2\left(5 + 7d^2\right) = d^2\left(12 + 7\left(d^2 - 1\right)\right) = 12d^2 + 7d^2\left(d^2 - 1\right) = 12d^2 + 7d\,(d+1)\,d\,(d-1)$$

$$2 \prec \begin{cases} d\,(d+1) \\ d\,(d-1) \end{cases} \Longrightarrow 4 \prec d\,(d+1)\,d\,(d-1)$$

$$3 \prec (d-1)\,d\,(d+1)$$

$$\Longrightarrow 12 = 3 \cdot 4 \prec d\,(d+1)\,d\,(d-1)$$